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The Propagation of Spherical Shock Waves
by Eric E. Ungar
AECU-3201

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THE PROPAGATION OF SPHERICAL SHOCK WAVES

Eric E. Ungar

ABSTRACT

A summary of unclassified theoretical work on propagation of one-dimensional shock waves and on the propagation of spherical shock waves in gases.

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I

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OBJECT: The purpose of this report is to present an investigation of one of the classic papers on one-dimensional shock waves in order to promote understanding of this phenomenon, and to present also an investigation of one of the modern theories of propagation of spherical shock waves.

SUMMARY: R. Becker's article "Stosswelle and Detonation" (1921) is examined in detail with respect to the theory of shock waves. The Hugoniot relations are developed and are shown to be independent of friction and heat transfer effects in the shock front.

The theory of propagation of spherical shocks, as developed by Kirkwood and Brinkley (1947) is discussed. Comparison of theoretical and experimentally measured quantities is shown to demonstrate the validity of the theory.

A glossary of hydrodynamical equations and a detailed annotated bibliography is also given.

Section I

ONE DIMENSIONAL SHOCK WAVES

INTRODUCTION

It seems that Poisson¹ was the first to find a simple wave solution of the differential equation of one-dimensional flow of a gas at constant temperature, and J. Challis² observed that such a solution equation can not always be solved uniquely for the velocity u .

To remedy this situation, Stokes³ proposed to assume a discontinuity in the velocity. He used the laws of conservation of mass and momentum to deduce two discontinuity conditions for an isothermal gas. Stokes was the first to suggest the possibility that discontinuous pressure waves might be propagated with velocities greater than that of sound. He argued further that discontinuities would never occur physically, because any tendency to form such a discontinuity would be counteracted by viscous forces.

S. Earnshaw⁴ first developed the laws of propagation of waves of finite amplitude (as distinguished from sound waves, which are assumed to be infinitesimal) and investigated mathematically the building up of a discontinuity. He realized from investigation of the flow of gases for which pressure is a function of density only, that since the local velocity of propagation increases across a compression wave, such a wave would always be "gaining" on its front until a discontinuity would form.

¹ S. D. Poisson - Memoire sur la theorie du son. Journal de l'ecole polytechnique, 14^{eme} cahier, 7, 319-392 (1808)

² J. Challis - On the velocity of sound. Phil-Mag. 32, 494-499 (1848)

³ G. G. Stokes - On a difficulty in the theory of sound. Phil-Mag. 33, 349-356 (1848)

⁴ S. Earnshaw - On the Mathematical Theory of Sound. Phil-Trans. A 150, 133-148 (1850)

B. Riemann⁵, without knowledge of Earnshaw's work, developed his own theory of the simple wave, and obtained the general solution to the flow problem by introducing what is now known as "Riemann invariants." He rediscovered the theory of shocks and elaborated on it, but incorrectly assumed the transition across a shock to be isentropic.

W. J. M. Rankine⁶ developed conditions for conservation of mass, momentum, and energy across the discontinuity. He showed that no steady adiabatic process in which the only forces are pressure-forces can represent a discontinuous change over a small finite region from one constant state to another. He proposed that across this region a non-adiabatic process occurs, rather than an adiabatic one, with heat being exchanged among the particles of the fluid, but none being received from outside it.

Although Rankine's work is compatible with the principle of conservation of energy, Rayleigh⁷ and Hugoniot⁸ were the first to point out clearly that isentropic transition in a shock would violate this principle. Hugoniot showed in fact, that for non-viscous flow without heat transfer (outside the discontinuity) entropy must be conserved for continuous flow, and must change across a discontinuity. Lord Rayleigh showed that dissipation is necessarily present in a shock wave, and that entropy across such a wave must hence increase.

⁵ B. Riemann - "Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite - Gesammelte Werke, 1876, p 144 or Abhandlungen der Gesellschaft der Wissenschaften zu Göttingen. Math-Phys Klasse 8, 43, (1860)

⁶ W. J. M. Rankine - One the thermodynamic theory of waves of finite longitudinal disturbance. Phil-Trans. A 160, 277-288 (1870)

⁷ J. Rayleigh - Aerial plane waves of finite amplitude Proc. Roy. Soc. A. 84, 247-284 (1910)

⁸ H. Hugoniot - Sur la propagation du mouvement dans les corps et spécialement dans les gaz parfaits. Journal de l'école polytechnique, 58, 1-125 (1889)

The earliest experimental work was done by Mach⁹ and his coworkers, who demonstrated by various methods that shock waves spread with velocities greater than that of sound.

The following discussion is a detailed theoretical investigation of one-dimensional shock waves and derives almost all of the results obtained by the earlier workers in the field by application of the principles of conservation of mass, momentum, and energy, and by ingenious mathematical procedures.

The investigation of one-dimensional shock waves is well justified in a paper which is supposed to be primarily concerned with spherical shocks. Not only is an understanding of one-dimensional shocks necessary for comprehension of spherical shocks, but in the treatment of one-dimensional shocks we are also able to reach some conclusions that apply to shock waves in general.

The following treatment follows that of Becker¹⁰ fairly closely, but additional mathematical steps and interpretations have been added, and comments by Lewis and von Elbe¹¹ have been included.

⁹ E. Mach, Wiener Berichte, 72(1875), 75(1877), 77(1878)

¹⁰ R. Becker - Stosswelle und Detonation. Zeits. f. Phys. 8, 321(1921)

¹¹ "Combustion, Flames, and explosion of Gases" B. Lewis and G. Von Elbe, Academic Press. Inc., New York, 1951

QUALITATIVE DISCUSSION OF SHOCK WAVE FORMATION

Before proceeding with a detailed mathematical analysis of shock wave formation, we might do well to become more acquainted with that phenomenon on a more qualitative basis.

Let us consider a long tube into which a piston is inserted at the left; the tube being filled uniformly with the gas of the surrounding atmosphere. If we impart a small velocity, dw , to the piston, this movement causes a weak compression wave to travel to the right with the velocity of sound. Then, at any given instant, the gas to the right of the wavefront is unchanged and undisturbed, while the gas between the wave-front and the piston is compressed an amount dp and has the velocity dw . If we now increase the velocity of the piston by another increment dw , a second weak compression wave will proceed to the right. By frequent repetition of this procedure, the piston is brought to its final velocity, w .

The compression waves that are started later advance with greater velocity since the velocity of sound is greater at higher densities (or at higher temperatures, if we assume the compression to be adiabatic), and since the gas itself also has higher flow velocity. Hence, as newer waves "catch-up" with the older ones, the wave-front becomes increasingly "steeper" - when older and newer waves are completely merged, a shock-wave with an extremely large pressure gradient is formed.

A MATHEMATICAL INVESTIGATION OF SHOCK WAVES

The Formation of Shock Waves

We consider a tube as described in the previous section, but with unit cross-sectional area. Let ξ denote the thickness of a very thin disk of the gas in the tube, let x be the space coordinate measured along the tube, and let t denote time. Then we see that any property G of a material particle along the tube is a function of x and t only. Hence we may write

$$(1) \quad \frac{dG}{dt} = \frac{\partial G}{\partial t} + u \frac{\partial G}{\partial x}$$

where $u = \frac{dx}{dt}$ denotes velocity along the tube axis. Further, we may show (see Appendix I) that

$$\frac{d\xi}{dt} = \xi \frac{\partial u}{\partial x}$$

Since we have defined ξ as very small, we may consider all properties constant within this thickness. If ρ denotes density, p pressure, E internal energy per unit mass, we note that ξ contains the mass $\rho \xi$, the momentum $u \rho \xi$, and the energy $\rho \xi (E + \frac{u^2}{2})$. Letting P_1 represent the mean pressure acting on the surfaces of our disk, and letting λ be a heat transfer coefficient, and μ one of friction, then from the principles of conservation of mass, momentum and energy, one may write the following:

$$(2a) \quad \frac{d}{dt} (\rho \xi) = 0$$

$$(2b) \quad \frac{d}{dt} (u \rho \xi) = - \frac{\partial P_u}{\partial x} \xi$$

$$(2c) \quad \frac{d}{dt} \left(\rho \xi \left[E + \frac{u^2}{2} \right] \right) = \xi \left[- \frac{\partial (P_u u)}{\partial x} + \frac{\partial}{\partial x} \left(\lambda \frac{\partial T}{\partial x} \right) \right]$$

where $P_u = P_1 - \mu \frac{\partial u}{\partial x}$ and where the relation between μ and the ordinary

viscosity coefficient η is $\eta = \frac{2}{3}\mu$ as dictated by the symmetric property of the pressure tensor.¹²

The above three relations are easily modified into

$$(3a) \quad \frac{dp}{dt} = -\rho \frac{\partial u}{\partial x}$$

$$(3b) \quad \frac{du}{dt} = -\frac{1}{\rho} \frac{\partial}{\partial x} (\rho - \mu \frac{\partial u}{\partial x})$$

$$(3c) \quad \frac{dE}{dt} = (\rho - \mu \frac{\partial u}{\partial x}) \frac{1}{\rho^2} \frac{dp}{dt} + \frac{1}{\rho} \frac{\partial}{\partial x} (\lambda \frac{\partial T}{\partial x})$$

The manipulations involved are shown in appendix II. If we introduce entropy s through the relation

$$T ds = dE + p dv = dE - p \frac{dp}{\rho^2}$$

then equation (3c) may be expressed as

$$(4) \quad \rho T \frac{ds}{dt} = \mu \left(\frac{\partial u}{\partial x} \right)^2 + \frac{\partial}{\partial x} \left(\lambda \frac{\partial T}{\partial x} \right)$$

which expresses the variation of s with time, under the influence of μ and λ , i.e. friction and heat conduction.

Now, let us neglect heat transfer and friction, for the sake of discussion.

This amounts to setting $\lambda = \mu = 0$, and hence, from equation (4), $\frac{ds}{dt} = 0$.

This, in turn implies that $s = \text{constant}$, or that the flow occurs adiabatically and reversibly.

Applying (1) to ρ , and using (3a) we find

$$(5a) \quad \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0$$

Similarly, applying (1) to u , and making use of (3b) after setting

$$\mu = 0,$$

¹² Weber and Gaus, Report. d. Phys. I, 1, p. 439

If we postulate a gas for which $p = p(\rho)$, (say an ideal gas for which $p = a^2 \rho^k$ where a^2 is a constant and k the ratio of heat capacities,) the above becomes

$$(5b) \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{dp}{d\rho} \frac{\partial \rho}{\partial x} = 0$$

Equations (5a) and (5b) may be solved as follows, to yield solutions of the form $u(x,t), \rho(x,t)$. We consider the x - t plane, in which the direction of any line element (dx, dt) may be given by φ , where φ is defined by $dx = \varphi dt$. Along this line element, any function $G(x,t)$ changes by the amount

$$dG = \left(\frac{\partial G}{\partial x} \varphi + \frac{\partial G}{\partial t} \right) dt$$

If G is chosen as $G = u + f(\rho)$, where f is any function of ρ only, with derivative f' , we find

$$d[u + f(\rho)] = \left(\frac{\partial u}{\partial x} \varphi + f' \frac{\partial \rho}{\partial x} \varphi + \frac{\partial u}{\partial t} + f' \frac{\partial \rho}{\partial t} \right) dt$$

By adding and subtracting $u \frac{\partial u}{\partial x} + f' u \frac{\partial \rho}{\partial x}$ within the parentheses, the above may be transformed into

$$d[u + f] = \left[\left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + [\varphi - u] f' \frac{\partial \rho}{\partial x} \right) + f' \left(\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \frac{\varphi - u}{f'} \frac{\partial u}{\partial x} \right) \right] dt$$

Referring to (5a) and (5b), we find that the right side of this expression vanishes if

$$(\varphi - u) f' = \frac{1}{\rho} \frac{dp}{d\rho} \quad \text{and} \quad \frac{\varphi - u}{f'} = \rho$$

that is, either if

$$f' = \frac{1}{\rho} \sqrt{\frac{dp}{d\rho}} \quad \text{and} \quad \varphi = u + \sqrt{\frac{dp}{d\rho}}$$

$$\text{or if} \quad f' = -\frac{1}{\rho} \sqrt{\frac{dp}{d\rho}} \quad \text{and} \quad \varphi = u - \sqrt{\frac{dp}{d\rho}}$$

We now define for ease of notation, $c^2 = \frac{dp}{d\rho}$. With this definition

the foregoing results may be written more concisely as

$$f' = \pm \frac{c}{\rho} \quad \text{and} \quad \varphi = u \pm c$$

But, since these are solutions which make $d(u + f) = 0$, we find that

$$(6a) \quad u + \int_{\rho_0}^{\rho} \frac{c}{\rho} d\rho = \text{Const. along curves} \quad \frac{dx}{dt} = u + c$$

and

$$(6b) \quad u - \int_{\rho_0}^{\rho} \frac{c}{\rho} d\rho = \text{Const. along curves} \quad \frac{dx}{dt} = u - c$$

ρ_0 here denotes a constant of integration, which is evaluated by application of initial conditions, and found to be the density of the undisturbed medium.

If we introduce Riemann's⁵ notation, to improve on Becker's cumbersome system, and define

$$\sigma = \int_{\rho_0}^{\rho} \frac{c(\rho)}{\rho} d\rho \quad \text{as indicated by R. H. Cole}^{13},$$

the above equations may be written more simply

$$u \pm \sigma = \text{Const. along curves} \quad \frac{dx}{dt} = u \pm c$$

Now we shall apply these results to the piston in our infinite tube. We assume the piston at rest at $x=0$ and $t=0$, let it be accelerated uniformly with an acceleration a for a time T and finally let it continue with constant velocity u_1 . If we denote quantities pertaining to the piston by the subscript s , the motion of the piston will be defined by

$$(7) \quad \begin{cases} x_s = \frac{1}{2} a t_s^2, & u_s = a t_s, & \text{for } 0 < t_s < T \\ x_s = a T t_s + \frac{1}{2} a T^2, & u_s = a T = u_1, & \text{for } t_s \geq T \end{cases}$$

This motion is shown as curve C in the x - t plane (Fig. 1).

In the entire tube, $u=0$, $\rho=\rho_0$ at time $t=0$. Since $u-\sigma$ then has the same value everywhere on the x -axis (at $t=0$), and since the curves (6b), originating from the positive x -axis, fill the entire space between C and the x -axis, then in that entire domain, $u-\sigma = \text{const.}$ Similarly, along the curves (6a), $u+\sigma = \text{const.}$ Hence, along the curves (6a) both u and σ must be constant, or both u and ρ must be

¹³ R. H. Cole - Underwater Explosions. Princeton University Press, Princeton, New Jersey, 1948

constant. ($\rho = \text{const.}$ implies $\sigma = \text{const.}$) But, along the x-axis, (i.e. at $t=0$), $u=0$, $\rho = \rho_0$. This means that along the x-axis $\sigma = 0$ also, and hence $u - \sigma = 0$, so that the relation between u and ρ that is valid everywhere is

$$(8) \quad u = \sigma = \int_{\rho_0}^{\rho} \frac{1}{\rho} \sqrt{\frac{d\rho}{d\rho}} d\rho$$

Immediately at the piston (i.e. along the curve C), u_s is given, and hence also ρ_s by the above equation. Then from every point (x_s, t_s) on C, the line

$$(9) \quad x - x_s = (t - t_s) \left(u_s + \left[\sqrt{\frac{d\rho}{d\rho}} \right]_s \right)$$

on which ρ and u have the constant values ρ_s and u_s , may be drawn.

For piston motion given by (7), the portion of the $x-t$ plane between the curve C and the x-axis is divided into three regions by the two lines (9) corresponding to $t_s=0$ and $t_s=T$. In the lowest region $u=0$, in the middle (shaded) region u varies from 0 to u_1 , and in the upper region $u = u_1$ (constant).

In a gas, where $p = a^2 \rho^k$, $\frac{dp}{d\rho} = a^2 k \rho^{k-1}$, whence

$$c = \sqrt{\frac{dp}{d\rho}} = a \sqrt{k} \rho^{\frac{k-1}{2}}$$

Also

$$\sigma = \int_{\rho_0}^{\rho} \frac{c}{\rho} d\rho = \int_{\rho_0}^{\rho} a \sqrt{k} \rho^{\frac{k-1}{2}-1} d\rho = \frac{2a\sqrt{k}}{k-1} \rho^{\frac{k-1}{2}} \Big|_{\rho_0}^{\rho}$$

If we now denote by c_0 the velocity of sound in the gas at the initial conditions, $c_0 = a \sqrt{k} \rho_0^{\frac{k-1}{2}}$, and we obtain from (8):

$$(10) \quad \begin{cases} u = \frac{2}{k-1} (c - c_0) & , \text{ and from this} \\ \frac{c}{c_0} = \left(\frac{\rho}{\rho_0} \right)^{\frac{k-1}{2}} = \left(\frac{\rho}{\rho_0} \right)^{\frac{k-1}{2k}} = 1 + \frac{u(k-1)}{2c_0} \end{cases}$$

The slopes of curves (6a) and (6b) then become

$$(11) \quad \begin{aligned} u + c &= u + \frac{k-1}{2} u + c_0 = c_0 + \frac{k+1}{2} u \\ u - c &= u - \frac{k-1}{2} u - c_0 = -(c_0 - \frac{3-k}{2} u) \end{aligned}$$

It is important to notice that the solutions we have obtained cease to be valid as soon as two of the lines (9) intersect, since at that point of intersection, u would be required to have two values at the same time. Hence the analytic counterpart of one wave-front over-taking the other is the intersection of two curves (9). The place X , and time

T of this over-taking are found by simultaneous solution of (9) with

$$-at_s = \frac{\kappa+1}{2} at - at_s(\kappa+1) - c_0$$

which is obtained by differentiating (9) with respect to t_s , after making use of (11) and (7) for replacing x_s , u_s , ζ_s , by functions of t_s . Then, we find

$$T = \frac{2}{\kappa+1} \left(\kappa t_s + \frac{c_0}{a} \right), \quad X = \frac{\kappa a}{2} t_s^2 + c_0 T$$

For $a = 200$ m/sec, $c_0 = 330$ m/sec, $\kappa = 1.4$, $T = 0.5$ sec, we find $u_s = 100$ m/sec. The first discontinuity will occur at $X = 453$ m, $T = 1.38$ sec (for $t_s = 0$).

Also, by (10), $\frac{P_1}{P_0} = 1.51$, $\frac{\rho_1}{\rho_0} = 1.34$

Fig. 2 corresponds to the above example. It represents the distribution of velocity along the axis of the tube at the times indicated. The "stiffening" of the wave-front is easily recognized by the increasing steepness of these curves. Although Becker gave a curve of this type, Fig. 2 is plotted from my computations. I plotted various lines of the type (9), making use of (7) and (11), in Fig. 3. From this figure, Fig. 2 is obtained easily by a cross-plot.

It is evident from the foregoing discussion, that the occurrence of a compression shock depends on the condition that within an adiabatic wave-formation the waves traveling in a denser medium tend to overtake those in a less dense medium. This means that the velocity given by

$$(6a) \quad \frac{dx}{dt} = u + \left(\sqrt{\frac{dp}{d\rho}} \right)_{\text{adiabatic}} \text{ must increase with increasing density.}$$

If we apply (8), the above condition becomes

$$\frac{d}{d\rho} \left(\int_{\rho_0}^{\rho} \frac{1}{\rho} \sqrt{\frac{dp}{d\rho}} d\rho + \sqrt{\frac{dp}{d\rho}} \right) > 0$$

But, the above expression may be written also as

$$\frac{1}{\rho} \sqrt{\frac{dp}{d\rho}} + \frac{d}{d\rho} \sqrt{\frac{dp}{d\rho}} = \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \sqrt{\frac{dp}{d\rho}} \right)$$

or, if we replace ρ by $\frac{1}{v}$, we obtain (as demonstrated in appendix

IV) that this expression may be written as

$$\frac{v^3}{2} \frac{d^2 p}{dv^2} \quad \text{Since } \frac{dp}{dv} < 0, \text{ and } v^3 > 0, \text{ the}$$

condition for the formation of compression shocks becomes

$$\left(\frac{d^2 p}{dv^2} \right)_{\text{adiabatic}} > 0$$

From the foregoing we draw the conclusion that in a given medium either

only compression or only rarefaction shocks can occur, not both, depend-

ing on whether $\left(\frac{d^2 p}{dv^2} \right)_{\text{adiabatic}}$ is positive or negative.

Necessity of Considering Heat Conduction and Friction

As mentioned before, the equations (5) are only applicable where heat transfer and viscosity may be neglected. Hence, the foregoing solutions obtained from these equations are subject to the same limitations. Since no real fluids are entirely devoid of these effects, equations (5) must yield wrong answers when temperature gradients $\left(\frac{\partial T}{\partial x} \right)$ or rate of change of volume $\left(\frac{\partial v}{\partial t} = -v^2 \frac{\partial \rho}{\partial t} \right)$ exceed certain limits. This is evident from inspection of equations (5). These magnitudes, however, tend to exceed any finite value in a shock wave, according to the foregoing discussion. Hence, use of equations (5) is possible only until the discontinuity occurs, but is not valid at the shock wave.

Intuitively, when the wave front reaches a certain steepness, a further increase in steepness will be prevented by friction and heat conduction; the flattening tendency will be just sufficient to counteract the tendency for steepening. The details of this interaction are too complicated for mathematical description, since various secondary waves are formed in the process. We hence must be satisfied with investigation of the shock wave after it has become quasi-steady.

Stationary Compression Shocks

After the shock wave has reached a quasi-steady state, we are able to investigate it using a coordinate system that moves with the shock. Our problem reduces to integration of equations (3) for the case where all derivatives with respect to time vanish. We hence replace $\frac{d}{dt}$ by $u \frac{d}{dx}$ and obtain the following equations from (3):

$$u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0$$

$$\rho u \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} (P - u \frac{\partial u}{\partial x}) = 0$$

$$u \frac{\partial E}{\partial x} = (P - u \frac{\partial u}{\partial x}) \frac{u}{\lambda} \frac{\partial \lambda}{\partial x} + \frac{\partial}{\partial x} (\lambda \frac{\partial T}{\partial x})$$

One may integrate these equations, as I have shown in appendix V, to obtain the following results.

$$(12a) \quad u = M v$$

$$(12b) \quad M^2 v + P - J = u M \frac{d\tau}{dx}$$

$$(12c) \quad E + J v - \frac{1}{2} M^2 v^2 - F = \frac{\lambda}{M} \frac{dT}{dx}$$

where M , J , F , are constants of integration, and energy E and temperature T may be considered as functions of r and τ .

We seek relations between the values u_1 , p_1 , τ_1 , before the shock front and the values u_2 , p_2 , τ_2 , behind it. This is easily accomplished when we realize that only within the wave front do $\frac{d\tau}{dx}$ and $\frac{dT}{dx}$ differ markedly from zero. Hence, for any place outside the wave-front

$$(13) \quad \begin{aligned} \frac{u}{v} &= M \\ M^2 v + P &= \frac{u^2}{v} + P = J \\ E + J v - \frac{1}{2} M^2 v^2 &= E + \frac{u^2}{2} + P v = F \end{aligned}$$

Comparing two such places, denoted by subscripts 1 and 2,

$$(14a) \quad \frac{u_1}{v_1} = \frac{u_2}{v_2}$$

$$(14b) \quad \frac{u_1^2}{v_1} + P_1 = \frac{u_2^2}{v_2} + P_2$$

$$(14c) \quad E_1 + \frac{u_1^2}{2} + P_1 v_1 = E_2 + \frac{u_2^2}{2} + P_2 v_2$$

These fundamental equations are hence independent of friction μ and heat transfer λ . They are identical to relations obtained by requiring conservation of mass, momentum, and energy, across the shock wave.¹⁴

Hence, the mathematical artifice of introducing a discontinuity is justified. (However, by this means we obtain no insight into the processes within the wave front).

Relations obtained for Compression Shocks

In order to obtain a clearer view of the meaning of equations (14), we solve (14a) and (14b) for u_1 , and u_2 , then substitute the results into (14c). We obtain

$$(15a) \quad u_1^2 = v_1^2 \frac{P_2 - P_1}{v_1 - v_2}$$

$$(15b) \quad u_2^2 = v_2^2 \frac{P_2 - P_1}{v_1 - v_2}$$

$$(15c) \quad E_2 - E_1 = \frac{1}{2} (P_1 + P_2) (v_1 - v_2)$$

Equation (15c) is known as the Hugoniot equation. Note that this is different from $E_2 - E_1 = \int_{v_1}^{v_2} p dv$, as would pertain to isentropic flow, but that for small enough differences of E and v the above approaches

$dE = p dv$, which is the isentropic relation. It may be shown¹⁴ that for the same volume change (15c) will read to a larger $\Delta E = E_2 - E_1$ than the corresponding isentropic relation.

The velocity of propagation, U , of the shock wave into a medium at rest, and W , the velocity of flow behind the wave front are

$$(16) \quad \begin{cases} U = u_1 = v_1 \sqrt{\frac{P_2 - P_1}{v_1 - v_2}} \\ W = u_1 - u_2 = (v_1 - v_2) \sqrt{\frac{P_2 - P_1}{v_1 - v_2}} \end{cases}$$

Hence, given the state of the undisturbed medium (P_1, v_1), and the "shock pressure", P_2 , plus an appropriate equation of state, we are able to compute all other values.

¹⁴ This is done by Lewis and Von Elbe, op. cit.

We illustrate this for an ideal gas, where

$$(17a) \quad p v = R T$$

and

$$(17b) \quad E_2 - E_1 = \bar{c}_v (T_2 - T_1) \quad \text{where } \bar{c}_v \text{ is the mean heat capacity between } T_2 \text{ and } T_1.$$

We let

$$(18) \quad \begin{cases} \zeta_1 = \frac{2 \bar{c}_v}{R} + 1 \\ \pi = P_2 / P_1 \end{cases}$$

and obtain (Appendix VI)

$$(19a) \quad \frac{T_2}{T_1} = \pi \frac{\pi + \zeta_1}{\pi \zeta_1 + 1}$$

$$(19b) \quad \frac{v_1}{v_2} = \frac{P_2}{P_1} = \frac{\pi \zeta_1 + 1}{\pi + \zeta_1}$$

$$(19c) \quad D^2 = P_1 v_1 \frac{\pi \zeta_1 + 1}{\zeta_1 - 1}$$

$$(19d) \quad W^2 = P_1 v_1 (\zeta_1 - 1) \frac{(\pi - 1)^2}{\pi \zeta_1 + 1}$$

For a perfect gas¹⁵, $c_v = \frac{R}{K-1}$, hence $\zeta_1 = \frac{K+1}{K-1}$. For a diatomic perfect gas¹⁵, $K = 1.4$, and hence $\zeta_1 = 6$. If π is large compared to 6, (19a) indicates that T increases proportionally to P . Hence, we must introduce ζ_1 , as a function of T .

Becker calculated the figures given in Table 1, using $\frac{273, T}{c_v} = 4.78 + .45 \times 10^{-3} T \frac{\text{cal}}{\text{mol}^\circ \text{K}}$ which he supposed to be good up to about 3000°C. (This expression, valid for O_2 and CO_2 , was taken from Pier, Zeits. f. Elektrochem. 15, 536 (1909) and 16, 897 (1910), and Siegel, Zeits. f. phys. Chem. 87, 641 (1914).) Although Lewis and Von Elbe state that this expression is somewhat in error, Becker's work still serves to point out relative magnitudes.

¹⁵ E. F. Obert. Thermodynamics, Mc-Graw-Hill Book Company, Inc. New York, 1948. p. 187, equ. (7-12)

With the above $\bar{\zeta}_2$, we obtain $\zeta_1 = 5.82 + 0.46 \times 10^{-3} T_2$, and with this ζ_1 , (19a) becomes a quadratic in T_2 . The other relations (19) then yield all other required values. $T_1 = 0^\circ\text{C}$.

TABLE I

π	$\frac{T_2}{T_1}$	$\frac{v_1}{v_2}$	T_2 (°K)	U (m/sec)	W (m/sec)	$\frac{i}{P_1}$	$(T_2)_{\text{adiab.}}$ (°K)
2	1.23	1.63	336	452	175	1.63	330
5	1.76	2.84	482	698	452	11.4	426
10	2.56	3.88	705	978	725	34.9	515
50	8.28	6.04	2260	2150	1795	296	794
100	14.15	7.66	3860	3020	2590	699	950
500	44.8	11.15	12200	6570	5980	5560	1433
1000	70.0	14.3	19100	9210	8560	14300	1710
2000	106.2	18.8	29000	12900	12210	37600	2070
3000	134.4	22.3	36700	15750	15050	66900	2180

The total impulse i , which appears in the next-to-last column, is an important measure for evaluating the effect of a shock wave on an obstacle. i is composed of the static pressure difference $P_2 - P_1$ and of the momentum of the gas behind the wave-front, $P_2 W^2$. With the value of W from (16), and by substituting $v_2 = \frac{1}{\rho_2}$,

$$i = (P_2 - P_1) + P_2 W^2 = (P_2 - P_1) \frac{v_1}{v_2}, \text{ and } \frac{i}{P_1} = (\pi - 1) \frac{v_1}{v_2}$$

In order to illustrate the foregoing calculations for a liquid, we use Tammann's equation of state¹⁶, which is valid up to very high pressures.

¹⁶ G. Tammann - Ann. Physik 37, 975 (1912)

It is

$$(20) \quad p = \frac{cT}{v-b} - K$$

where c , b , and K , are constants. The general relation for internal energy, $dE = c_v dT + (T \frac{\partial p}{\partial T} - p) dv$ then may be integrated to yield

$$(20a) \quad E = \bar{c}_v T + K v$$

If we define

$$(21) \quad p' = p + K, \quad v' = v - b$$

and apply (20) to the Hugoniot equation (15c), we obtain

$$\bar{c}_v (T_2 - T_1) = \frac{1}{2} (p' + p'_2) (v'_1 - v'_2)$$

and (20) is transformed into

$$p' v' = c T$$

The above equations are structurally identical to those for a gas, so that the solutions (19) can be carried over directly. Here,

$$\xi_1 = \frac{2 \bar{c}_v}{c} + 1, \quad \text{and} \quad \pi' = \frac{p_2 + K}{p_1 + K}, \quad \text{making}$$

$$(22) \quad \frac{T_2}{T_1} = \pi' \frac{\pi' + \xi_1}{\pi' \xi_1 + 1}, \quad \frac{v'_1 - b}{v'_2 - b} = \frac{\pi' + \xi_1}{\pi' \xi_1 + 1}, \quad D^2 = v'_1^2 \frac{p_1 + K}{v'_1 - b} \frac{\pi' \xi_1 + 1}{\xi_1 - 1}$$

Table II applies to ethyl ether, for which $K = 2729 \text{ atm}$, $c = .1001 \frac{\text{cal}}{\text{g}^\circ\text{K}}$, $b = .94 \frac{\text{cm}^3}{\text{g}}$, $v_1 = 1.36 \frac{\text{cm}^3}{\text{g}}$, $\bar{c}_p = .564 \frac{\text{cal}}{\text{g}^\circ\text{K}}$, $p_1 = 1 \text{ atm}$, $T_1 = 0^\circ\text{C}$

TABLE II			
$p_2 \text{ (atm)}$	$(T_2 - T_1)_{\text{adiab}}$	$(T_2 - T_1)_{\text{shock-w.}}$	$D \text{ (m/sec)}$
100	1.6	1.6	1260
1,000	15.6	15.6	1445
10,000	85	113	2680
20,000	123	211	3000
60,000	201	594	5010
100,000	245	975	6430

Structure of the Shock Wave

Becker investigated the structure of the wave-front by carrying out the integration of equations (12) for gases in which λ and μ fulfill certain requirements. The usefulness of his calculations, of course, depends on how well a gas under consideration meets these requirements. For air, Becker admitted an error of about 10% in his calculations for v , p , T , in the shock front, but he computed thicknesses of shock fronts on the order of magnitude of molecular mean free paths. However, for such magnitudes, the physics of continua (hydrodynamics) can not apply, and Becker admitted his theory was inadequate.

Becker obtained the following shock front thicknesses δ , for air with $p_1 = 1$ atm, $T_1 = 0^\circ\text{C}$

p_2 (atm)	2	5	10	100	1000	2000	3000
$10^7 \delta$ (cm)	447	117	66	16.5	5.2	3.6	2.9

(According to Kinetic theory of gases, the mean free path at 1 atm, 0°C is about 90×10^{-7} cm, and the mean distance between molecules is 3.3×10^{-7} cm).

But Becker's theory was vindicated by L. H. Thomas¹⁷ in 1944. He showed that all shock waves in air are a few mean free paths thick, by considering the increase in the coefficients of thermal conductivity and viscosity with increasing temperature and pressure, which Becker neglected, and by application of Kinetic theory.

Experimental substantiation of theories of processes within a shock front is practically nil. The extremely transient nature of shock front passage

¹⁷ L. H. Thomas - Note on Becker's Theory of the Shock Front. J. Chem. Phys., 12, pp 449-57 (November 1944)

makes instrumentation a formidable problem, and besides makes knowledge of conditions within the shock front of little practical significance.

G. R. Cowan and D. F. Hornig¹⁸ developed equations for computation of the density profile of a shock wave from measurements of reflectivity. They also performed some measurements in nitrogen, keeping a constant $P_2/P_1 = 1.71$, and varying P_1 . The following are their results:

P_1 (atm)	5.78	4.63	2.86
l (10^{-7} cm)	180	200	320

The accuracy of their measurements is estimated at 25%; hence the above values are slightly larger than those calculated by the Becker - Thomas Theory. This discrepancy may be due to the assumption of hard-sphere molecules made by Thomas, which is not too good for a diatomic gas like nitrogen. But assumptions made by Cowan and Hornig may possibly lead to a greater error in their work than was estimated.

¹⁸ G. R. Cowan and D. F. Hornig, The Experimental Determination of the Thickness of a Shock front in a Gas.
J. Chem. Phys. 18, 1008-18 (August 1950)

Section II

THE PROPAGATION OF SPHERICAL SHOCK WAVES

INTRODUCTION

Although Lord Rayleigh⁷ solved the hydrodynamic equations for plane shocks analytically (assuming adiabatic relations between pressure and density), such an approach can not be taken in the case of spherical shock waves. The difficulty arises from the spherical divergence terms that appear in the partial differential equations that describe spherical motion.

A solution of the problem may of course be obtained by numerical integration of the pertinent partial differential equations. Penney²⁰, and later Penney and Dasgupta²¹ have carried out calculations of this kind for spherical TNT charges. They employed a method of integration based on the equations of Riemann⁵, which greatly simplifies the numerical work and reduces the required labor. But, even with this simplification, this method is extremely tedious and complex. Furthermore, the Riemann equations apply only where dissipation effects can be neglected. The desirability of developing more rapid and flexible methods is hence evident.

J. G. Kirkwood and H. A. Bethe²² developed one such method, applicable to underwater explosions of spherical charges. However, they used the approximation of adiabatic flow, neglecting the increase in entropy at the shock front. This assumption is warranted for water, because the entropy increment produced by a shock wave in liquids is very small. However, for gases the entropy increase is not negligible.

²⁰ W. G. Penney, British Report RC - 142 (1941)

²¹ W. G. Penney and H. K. Dasgupta, British Report RC - 333 (1942)

²² J. G. Kirkwood and H. A. Bethe. - Pressure waves produced by an underwater explosion, I, OSRD 588 (1942)

Another theory of underwater shock waves has been presented by Osborne and Taylor²³. This theory, however, is based upon the acoustic approximation, and is hence strictly valid only for small excess pressures at relatively large distances from the source. This approximation is also not valid for very large distances from the source, as indicated by G. B. Whitham²⁴.

The following theory, which we shall examine in detail, was developed by S. R. Brinkley, Jr., and J. G. Kirkwood²⁵. It is superior to those discussed above, and to others developed for shock waves in air, which have been based on ideal gas adiabatics with constant heat capacity²⁶. Brinkley and Kirkwood account in their theory for the finite entropy increment occurring in the fluid due to passage of the shock wave. Hence, this theory is equally valid for the propagation of shock waves in liquids and in gases.

The development which follows is essentially that of Brinkley and Kirkwood; however, I have specialized it for application to spherical shocks (since their discussion applies to plane and cylindrical flow also) and I have inserted additional explanation and intermediate mathematical steps where these seemed applicable.

²³ F. M. Osborne and A. H. Taylor. - Nonlinear propagation of underwater shock waves. Phys. Rev. 70, 322 - 8 (September 1, 15, 1946)

²⁴ G. B. Whitham. - The propagation of spherical blast. Proc. Roy. Soc. A, 203, 571 - 81 (October 24, 1950)

²⁵ S. R. Brinkley, Jr., and J. G. Kirkwood. - Theory of the Propagation of Shock Waves. Phys. Rev. 71, 606 - 11 (May 1, 1947)

THE BRINKLEY AND KIRKWOOD THEORY

Brinkley and Kirkwood use a hybrid form of the hydrodynamic equations, utilizing the Euler form of these equations, but introducing Lagrangian Coordinates²⁷. In order to derive the equations as they are used in this theory, we define the following symbols.

Let r be the Euler coordinate at time t of an element of fluid with Lagrange coordinate R . Let p and ρ denote pressure and density, and let the subscript (0) refer to the undisturbed fluid. The Euler velocity of sound c , is given by $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$, and let $P = p - p_0$.

The equation of motion (conservation of momentum) in the Euler form, may be written in vector notation²⁷ as

$$(23) \quad \rho \frac{d\mathbf{v}}{dt} + \nabla p = 0, \text{ where } \mathbf{v} \text{ is the velocity vector.}$$

For spherical flow, all quantities are functions only of radius r and time t , and the velocity is purely radial. For this type of flow, (23) becomes

$$(24) \quad \rho \frac{du}{dt} + \left(\frac{\partial p}{\partial r}\right)_t = 0$$

where u is the particle velocity, or, $u = \left(\frac{\partial r}{\partial t}\right)_R$

But $u = u(R, t)$, and Euler's notation means $R = \text{const.}$

Hence

$$\frac{du}{dt} = \left(\frac{\partial u}{\partial R}\right)\left(\frac{\partial R}{\partial t}\right)_R + \left(\frac{\partial u}{\partial t}\right)_R = \left(\frac{\partial u}{\partial t}\right)_R$$

We note that a spherical shell with radius R and thickness dR contains a mass of fluid given by $4\pi\rho_0 R^2 dR$, (R is by definition the position in the undisturbed fluid of a volume element which at time t is at r), and that the same mass of fluid is contained in a spherical shell

²⁶ G. I. Taylor The air wave surrounding an expanding sphere. Proc. Roy. Soc A, 186, 273 - 92 (September 24, 1946) - This assumption fails badly near the explosive source.

²⁷ See the "Glossary of Hydrodynamic Equations" near the end of this paper.

of radius r and thickness dr , which contains a mass $4\pi\rho r^2 dr$.

Hence, at time t , $4\pi\rho_0 R^2 dR = 4\pi\rho r^2 dr$, from

which we find $\left(\frac{\partial r}{\partial R}\right)_t = \frac{\rho_0 R^2}{\rho r^2}$

Then, considering $\rho = \rho(R, t)$

$$\left(\frac{\partial \rho}{\partial r}\right)_t = \left(\frac{\partial \rho}{\partial R}\right)_t \left(\frac{\partial R}{\partial r}\right)_t + \left(\frac{\partial \rho}{\partial t}\right)_R \left(\frac{\partial t}{\partial R}\right)_t = \left(\frac{\partial \rho}{\partial R}\right)_t \frac{\rho r^2}{\rho_0 R^2}$$

Substituting the foregoing values into (24), we obtain

$$\rho \left(\frac{\partial u}{\partial t}\right)_R + \frac{\rho r^2}{\rho_0 R^2} \left(\frac{\partial \rho}{\partial R}\right)_t = 0$$

or, as in the paper by Brinkley and Kirkwood,

$$(25a) \quad \frac{R^2}{r^2} \left(\frac{\partial u}{\partial t}\right)_R + \frac{1}{\rho_0} \left(\frac{\partial \rho}{\partial R}\right)_t = 0$$

since $d\rho = dp$

Euler's equation of continuity in vector notation appears as

$$(26) \quad \frac{d\rho}{dt} + \rho (\nabla \cdot \mathbf{v}) = 0$$

Again, constant R is implied. Considering $\rho = \rho(R, t)$ again,

then

$$\frac{d\rho}{dt} = \frac{d\rho}{dR} \frac{dR}{dt} = \frac{1}{c^2} \frac{d\rho}{dt} = \frac{1}{c^2} \left[\left(\frac{\partial \rho}{\partial R}\right)_t \left(\frac{\partial R}{\partial t}\right)_R + \left(\frac{\partial \rho}{\partial t}\right)_R \right] = \frac{1}{c^2} \left(\frac{\partial \rho}{\partial t}\right)_R$$

Further, for spherical flow, and considering $u = u(r, t)$,

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u)$$

by the definition of ∇ , for curvilinear coordinates²⁸; or

$$\begin{aligned} \nabla \cdot \mathbf{v} &= \left(\frac{\partial u}{\partial r}\right)_t + \frac{2u}{r} = \left(\frac{\partial u}{\partial R}\right)_t \left(\frac{\partial R}{\partial r}\right)_t + \frac{2u}{r} \\ &= \frac{\rho r^2}{\rho_0 R^2} \left(\frac{\partial u}{\partial R}\right)_t \end{aligned}$$

²⁸ See for instance, H. Lass, "Vector and Tensor Analysis" McGraw-Hill Book Co., Inc., New York, 1950. p 54, equ. (89)

Substitution of the foregoing results into (26), and division by ρ yields Kirkwood and Brinkley's form of the equation of continuity:

$$(25b) \quad \frac{1}{\rho c^2} \left(\frac{\partial P}{\partial t} \right)_R + \frac{zu}{r} + \frac{\rho r^2}{\rho_0 R^2} \left(\frac{\partial u}{\partial R} \right)_t = 0$$

Summarizing, we have the new equation of motion

$$(25a) \quad \frac{R^2}{r^2} \left(\frac{\partial u}{\partial t} \right)_R + \frac{1}{\rho_0} \left(\frac{\partial P}{\partial R} \right)_t = 0$$

and the equation of continuity,

$$(25b) \quad \frac{1}{\rho c^2} \left(\frac{\partial P}{\partial t} \right)_R + \frac{\rho r^2}{\rho_0 R^2} \left(\frac{\partial u}{\partial R} \right)_t + \frac{zu}{r} = 0$$

Equations (25) are to be solved subject to the initial conditions specified on a curve in the R, t -plane and to the Hugoniot conditions at the shock front (Appendix VII):

$$(26) \quad \begin{aligned} P &= \rho_0 u U \\ \rho(U-u) &= \rho_0 U \\ \Delta H &= \frac{1}{2} P \left(\frac{1}{\rho_0} + \frac{1}{\rho} \right) \end{aligned}$$

ΔH here denotes the specific enthalpy increment experienced by the the fluid in traversing the shock front, and U is the velocity of the shock front. (The Hugoniot conditions are seen to apply to spherical shock fronts also, since in their derivation no assumption as to the shape of the fronts was made).

At the shock-front itself, $r = R$, so that equations (25) for the shock front simplify to

$$(25c) \quad \left(\frac{\partial u}{\partial t} \right)_R + \frac{1}{\rho_0} \left(\frac{\partial P}{\partial R} \right)_t = 0$$

$$(25d) \quad \frac{P}{\rho_0} \left(\frac{\partial u}{\partial R} \right)_t + \frac{1}{\rho c^2} \left(\frac{\partial P}{\partial t} \right)_R + \frac{zu}{R} = 0$$

We define now a derivative in which the shock front is stationary by the operator

$$\frac{d}{dR} = \frac{\partial}{\partial R} + \frac{\partial t}{\partial R} \cdot \frac{\partial}{\partial t} = \frac{\partial}{\partial R} + \frac{1}{U} \frac{\partial}{\partial t}$$

and apply this operator to the first of equations (26).

After transformation we obtain (Appendix VIII)

$$(27) \quad \frac{\partial u}{\partial t} + U \frac{\partial u}{\partial R} - \frac{g}{\rho_0} \frac{\partial P}{\partial R} - \frac{g}{\rho_0 U} \frac{\partial P}{\partial t} = 0$$

where
$$g = \rho_0 U \frac{du}{dP} = 1 - \frac{P}{U} \frac{dU}{dP}$$

We note that all of the coefficients of equations (25c), (25d), and (27) may be expressed as functions of pressure alone by means of the Hugoniot conditions (26) and an appropriate equation of state of the fluid. If we can find a fourth relation between the four partial derivatives, we would be able to solve for each of the four derivatives as a function of P and R and to formulate by means of the operator we defined previously, an ordinary differential equation

$$(28) \quad \frac{dP}{dR} = \frac{\partial P}{\partial R} + \frac{1}{U} \frac{\partial P}{\partial t} = F(P, R)$$

for the peak pressure P of the shock wave as a function of the distance R . (Henceforth we shall use u' and P' to denote particle velocity and excess pressure behind the shock front, and reserve the unprimed letters for these quantities at the shock front).

The physical basis for establishing the desired fourth relation is the fact that as a shock wave passes through a fluid, it leaves in its path a residual internal energy increment in each element of the fluid. This energy increment is determined by an entropy change produced by passage of the shock front. As a result, the energy propagated by the shock wave decreases with distance from the source.

Clearly, the work done by the shock wave after it passes a point R is ultimately transformed into internal energy. Consider the work W done by the source of initial radius α . This work may be resolved into two parts: that which is transformed into internal energy of the fluid at a pressure p_0 ²⁹ within a sphere of radius R , and that work which is

²⁹ Note that passage of a shock front through a fluid element leaves it at increased entropy and internal energy, from which it returns to the initial pressure p_0 . Absence of heat transfer is assumed throughout.

done on this spherical surface by the external fluid.

The first part of this work, W_{01} , may be found by integrating E between the initial and final spherical surfaces. The mass dm contained in a spherical shell of radius r_0 , and thickness dr_0 , is $dm = 4\pi\rho_0 r_0^2 dr_0$. ρ_0 is used here because the fluid is initially in the undisturbed state.

Hence,

$$W_{01} = \int E dm = 4\pi \int_{a_0}^R E \rho_0 r_0^2 dr_0$$

This integral may

be evaluated if $E = E(P)$, is known; P being the peak pressure, and if $P = P(r_0)$ can be found.

The second part of the work, W_{02} , may be found from the definition of mechanical work, as the integral of normal force times distance. Here, our force is pressure times area of a spherical surface, and

$$W_{02} = 4\pi \int_{t_0(R)}^{\infty} r^2 (P' + p_0) u' dt^{30}$$

Hence,

$$(29) \quad \frac{W_0}{4\pi} = \int_{a_0}^R E \rho_0 r_0^2 dr_0 + \int_{t_0(R)}^{\infty} r^2 (P' + p_0) u' dt$$

The term involving p_0 in the time integral gives the product of p_0 and the volume displacement of the fluid element initially at R . This displacement is obviously the sum of the outward volume displacement ΔV_3 of the inner boundary of the fluid (i.e., the expansion of the explosion products) and the displacement of the volume of fluid initially between shells of radii a_0 and R to shells of radii a' and R' . We may hence write

$$4\pi \int_{t_0(R)}^{\infty} p_0 r^2 u' dt = p_0 \Delta V_3 + 4\pi p_0 \left[\int_{a'}^{R'} r^2 dr - \int_{a_0}^R r^2 dr \right]$$

³⁰ Note that $p' = P' / p_0$, and $u' dt$ is the displacement of a fluid element in time dt ; $t_0(R)$ is the time of arrival of the shock front at R .

or

$$4\pi \int_{t_0(R)}^{\infty} \rho_0 r^2 u dt = \rho_0 \Delta V_g + 4\pi \rho_0 \int_{a_0}^R \left(\frac{\rho_0}{\rho} - 1 \right) r_0^2 dr_0$$

from the relation $\rho r^2 dr = \rho_0 r_0^2 dr_0$ between Euler and Lagrange coordinates as previously derived.

If we now introduce $h = E / \rho_0 \Delta(\frac{1}{\rho})$, the specific enthalpy increment, we find

$$\begin{aligned} \int_{a_0}^R E \rho_0 r_0^2 dr_0 &= \int_{a_0}^R h \rho_0 r_0^2 dr_0 - \int_{a_0}^R \rho_0 \rho_0 r_0^2 \left(\frac{1}{\rho} - \frac{1}{\rho_0} \right) dr_0 \\ &= \int_{a_0}^R h \rho_0 r_0^2 dr_0 - \int_{a_0}^R \rho_0 \left(\frac{\rho_0}{\rho} - 1 \right) r_0^2 dr_0 \end{aligned}$$

Substituting then, (29) becomes

$$(30) \quad \frac{W_0}{4\pi} = \int_{a_0}^R h \rho_0 r_0^2 dr_0 + \frac{\rho_0 \Delta V_g}{4\pi} + \int_{t_0(R)}^{\infty} r^2 P' u' dt$$

The time integral is assumed to vanish as the shock front passes to infinite distance. Subtracting from (30) its value at $R = \infty$, we obtain

$$0 = \int_{a_0}^R h \rho_0 r_0^2 dr_0 - \int_{a_0}^{\infty} h \rho_0 r_0^2 dr_0 + \int_{t_0(R)}^{\infty} r^2 P' u' dt$$

from which

$$(31) \quad \int_R^{\infty} h \rho_0 r_0^2 dr_0 = \int_{t_0(R)}^{\infty} r^2 P' u' dt = D(R)$$

The energy of the shock wave at the point R is by definition the work done on the fluid exterior to R . (Since total energy of shock wave is energy expended from source to R plus energy expended outside R , the energy remaining in the shock wave when it is at R is the total energy minus that expended between the source and R , or the energy expended outside R). The shock wave energy at R is hence given by $4\pi D(R)$,

$D(R)$ as written above does, however, depend on the size of the source, and it is hence desirable to normalize the time integral to a value which is independent of this factor. This is done by expressing the integrand as a fraction of its initial value $R^2 P u$, and choosing a reduced time scale

for which the initial slope of the integrand has unit value. Inasmuch as the observed energy-time curves are approximately of the form $R^2 P u e^{-\gamma u}$, it is convenient to use as a time unit the initial logarithmic slope of the curve, defined by

$$\frac{1}{\mu} = - \frac{\partial}{\partial t} (\log P' u' r^2)_{t=t_0(R)}$$

After the differentiation is carried out, this becomes

$$(32) \quad \frac{1}{\mu} = - \frac{1}{P} \frac{\partial P}{\partial t} - \frac{1}{u} \frac{\partial u}{\partial t} - \frac{2u}{R}$$

Then, $D(R)$ may be expressed as

$$(33) \quad D(R) = R^2 P u \mu \nu$$

Where $\nu = \nu(R) = \int_0^\infty f(R, \tau) d\tau$, τ being given by

$$\tau = \frac{t - t_0(R)}{\mu} \quad \text{and} \quad f(R, \tau) = \frac{r^2 P' u'}{R^2 P u}$$

Elimination of μ between (32) and (33) yields the desired fourth relation between the partial derivatives at the shock front:

$$(34) \quad \frac{R^2 P u \nu}{D(R)} = - \frac{1}{P} \frac{\partial P}{\partial t} - \frac{1}{u} \frac{\partial u}{\partial t} - \frac{2u}{R}$$

The set of equations (25c), (25d), (27), and (34) is exact, involving integrals of equations (25a) and (25b) for knowledge of the reduced energy-time function $f(R, \tau)$.

However, if $f(a, \tau)$ is initially a monotone decreasing function of τ , $f(R, \tau)$ will remain so, and in fact will at large R become asymptotically a quadratic function of τ corresponding to the linear form of the pressure-time curve that has been shown to be stable at large distances³¹.

³¹ J. G. Kirkwood and H. A. Bethe. OSRD 588(1942)

This means that ν is a very slowly varying function of R , for which sufficiently accurate estimates for many purposes may be made without explicit integration of the hydrodynamic equations. The assignment of a value which is independent of R to ν is equivalent to imposing a similarity restraint on the energy-time curve. This type of restraint has been used before, and has proven to be a very reasonable assumption.^{26,32}

The initial energy-time and pressure-time curves of an explosion wave are rapidly decreasing. For an initial estimate of ν , an expansion of the logarithm of the function in a Taylor series in time is appropriate. (The well-known peak-approximation). This corresponds to an exponential $f(\tau) = e^{-\tau}$, which gives the result $\nu = \int_0^\infty e^{-\tau} d\tau = 1$. For the asymptotic quadratic energy-time curve, $f(\tau) = (1 - \tau/2)^2$ for $\tau \leq 2$ and $f(\tau) = 0$ for $\tau > 2$, we find $\nu = \int_0^2 (1 - \tau/2)^2 d\tau = \frac{2}{3}$.

As a convenient empirical interpolation formula between these two values, Kirkwood and Brinkley have used, in a different paper.³³

$$\nu = 1 - \frac{1}{3} e^{-P/R}$$

in a series of calculations of the peak pressure distance curves for spherical shock waves.

Equations (25c), (25d), (27), and (34) may be solved simultaneously for the four partial derivatives, and an ordinary differential equation for P as a function of R may be formulated with the aid of equ. (28).

³² G. Taylor - Similarity solutions to problems involving gas flow and shock waves. Proc. Roy. Soc. A. 204, p 8-9, (November 22, 1950)

³³ J. G. Kirkwood and S. R. Brinkley, Jr., OSRD 4814 (1945)

A second ordinary differential equation relating D to R may be obtained by differentiation of (31).

The results of the above manipulations (as indicated in appendix X), may be written in the following form.

$$(35) \quad \frac{dD}{dR} = -R^2 L(P)$$

$$\frac{dP}{dR} = -v \frac{R^2 P^3}{D} M(P) - \frac{P}{R} N(P)$$

where

$$L(P) = \rho_0 h(P)$$

$$M(P) = \frac{1}{\rho_0 U^2} \frac{G}{Z(1+g) - G}$$

$$N(P) = \frac{4(P_0/\rho) + 2G(1 - P_0/\rho)}{2(1+g) - G}$$

$$G = 1 - \left(\frac{\rho_0 U}{\rho C}\right)^2$$

$$g = 1 - \frac{P}{U} \frac{dU}{dP}$$

The functions $L(P)$, $M(P)$, $N(P)$ can be evaluated as functions of the pressure by means of the equation of state of the fluid and the Hugoniot relations (26). (This is illustrated for the case of a perfect gas, in appendix XI). The equations (35) may then be integrated numerically by the use of standard methods³⁴, provided the initial conditions are known.

The constants of integration may be determined in one of two ways:

- (1) by the initial conditions at the boundary of the explosion products following detonation, or
- (2) by an experimental pressure-time curve at a selected distance.

The first method is a theoretical one, and is based on the thermo-chemistry connecting the explosive and its products. The study of detonation is a field

³⁴ See, for instance, J. B. Scarborough, "Numerical Mathematical Analysis." Johns Hopkins Press, Baltimore, Md., 1930, pp218 ff.

in itself, and is hence beyond the scope of this paper. Some discussion of this may be found in R. H. Cole's book "Underwater Explosions", and Brinkley and Kirkwood have published a paper³⁵ giving tables and graphs for application of this type of boundary condition to numerical integration. According to Cole, the most convenient initial constants in this case are the initial pressure and either the parameter μ or the total shock-wave energy.

If the semi-empirical approach is used, the natural constants of integration to be chosen are the initial peak pressure and the energy integral. However, Cole mentions that it is difficult to evaluate the energy integral very accurately from experimental pressure-time curves. Hence, experimental data of peak pressure as a function of distance are often used instead, for the determination of initial conditions from the equation for $\frac{dP}{dR}$.

I have not worked out a numerical example here since the computational methods are well known³⁴, and since relatively little would be gained by such an example compared to the time required for computation. (Note the complexity of the equations derived in Appendix XI).

³⁵ OSRD 1231 (1943). Calculation of detonation pressures for several types of explosives.

COMPARISON OF EXPERIMENTAL DATA WITH VALUES CALCULATED
BY THE THEORY OF KIRKWOOD AND BRINKLEY

In Fig. 4. I have plotted the overpressure ratio $\pi = \frac{P-P_0}{P_0}$ against reduced distance $Z = \frac{R}{W^{1/3}}$. The reason for this type of graph is based on the law of similarity³⁶, which asserts that pressure and other properties of a shock wave will be unchanged if the scales of length and time by which it is measured are changed by the same factor as the dimensions of the charge. Since the weight of the charge is a constant (density) times its volume, multiplying the weight by a factor f amounts to multiplying each linear dimension of the charge by $f^{1/3}$. Hence, the expression Z for reduced distance is constructed to make all properties as functions of Z independent of charge weight for a given explosive. For TNT, the density is such that it makes $Z = 1.06 \frac{R}{d}$, where d is the diameter of a sphere having the same volume as the charge, and Z is thus approximately the distance from the explosive source in charge diameters³⁷.

Fig. 4. shows a theoretical curve for explosion of TNT spheres, as computed by the method of Kirkwood and Brinkley by the originators of the method. Unfortunately, I was unable to find data which are exactly comparable to the computed curve.

³⁶ First stated by H. W. Hilliar, (British) department of Scientific Research and Experiment report RE 142/19 (1919)

³⁷ R. G. Stoner and B. Bleakney. "The Attenuation of spherical shock waves in air." J. Appl. Phys. 19, 670 -678 (July, 1948)

Instead, the following data are shown plotted in Fig. 4.

Charges	Experimenters
3 3/4 lb pentolite spheres	Stoner and Bleakney ³⁷
Cylinders of TNT/CE, ranging between 3 and 76 lb.	Grime and Sheard ³⁸

The curve for the pentolite spheres is seen to approximate the theoretical curve for TNT spheres very closely, and lies slightly above it throughout the plotted range. This discrepancy may be explained by the difference in the explosives, pentolite having approximately a 5% superiority in pressure level over TNT, according to the experimenters.

The plot of data obtained for TNT/CE is seen to be similar in shape to the computed curve, but to lie a good deal above it. This difference may be explained by superiority of TNT/CE to ordinary TNT, and by the fact that the experimental charges were cylindrical (with diameter equal to height) whereas the computed curves applies to spherical charges. Further, the measurements were taken near the ground, which may result in some distortion of experimental results.

As far as we may conclude from the above discussion, the theory of Brinkley and Kirkwood thus seems to be in good agreement with experimentally obtained values.

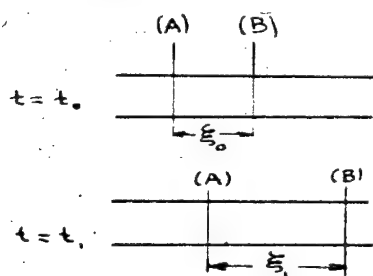
38

G. Grime and H. Sheard. "The experimental study of blast from bombs and bare charges." Proc. Roy. Soc. A 187, 357 (1946)

APPENDIX I

Proof of $\frac{d\xi}{dt} = \xi \frac{\partial u}{\partial x}$

We investigate what happens to a thin disk of material in our tube during a small time interval by considering the boundaries of the disk. ξ denotes the thickness of the disk, and the subscripts 0 and 1 refer respectively to the instants at the beginning and end of the time interval, $t_1 - t_0$. For convenience, we designate the boundaries of the disk by (A) and (B).



At $t = t_0$, (A) is at the coordinate x , and (B) at $x + \xi_0$; (A) is moving with a velocity u and hence (B) moves with a velocity $u + \frac{\partial u}{\partial x} \xi_0$. If we denote $t_1 - t_0$ by Δt , then at t_1 , (A) will be at $x + u \Delta t + \frac{\partial u}{\partial t} (\Delta t)^2$, and (B) will have moved to

$x + \xi_0 + (u + \frac{\partial u}{\partial x} \xi_0 + \frac{\partial u}{\partial t} \Delta t) \Delta t$. To find ξ_1 we now subtract the coordinate of (A) at t_1 from that of (B) at the same time; or

$$\begin{aligned} \xi_1 &= x + \xi_0 + u \Delta t + \frac{\partial u}{\partial x} \xi_0 \Delta t + \frac{\partial u}{\partial t} (\Delta t)^2 \\ &\quad - x - u \Delta t - \frac{\partial u}{\partial t} (\Delta t)^2 \\ &= \xi_0 + \frac{\partial u}{\partial x} \xi_0 \Delta t \end{aligned}$$

Hence $\Delta \xi = \xi_1 - \xi_0 = \frac{\partial u}{\partial x} \xi_0 \Delta t$
and

$$\frac{d\xi}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \xi}{\Delta t} = \frac{\partial u}{\partial x} \xi$$

APPENDIX II

Conversion of Equations (2) to Equations (3)

After we carry out the differentiation indicated in

$$(2a) \quad \frac{d}{dt} (\rho \xi) = 0$$

we obtain

$$\frac{d\rho}{dt} \xi + \rho \frac{d\xi}{dt} = 0$$

We have shown that $\frac{d\xi}{dt} = \xi \frac{\partial u}{\partial x}$. With this substitution, and after dividing by $\xi \neq 0$, we find

$$(3a) \quad \frac{d\rho}{dt} + \rho \frac{\partial u}{\partial x} = 0 \quad \text{or} \quad \frac{d\rho}{dt} = -\rho \frac{\partial u}{\partial x}$$

Similarly, after we carry out the differentiation indicated in

$$(2b) \quad \frac{d}{dt} (u \rho \xi) = -\frac{\partial p}{\partial x} \xi$$

we find

$$\rho \xi \frac{du}{dt} + u \left(\xi \frac{d\rho}{dt} + \rho \frac{d\xi}{dt} \right) = -\frac{\partial p}{\partial x} \xi$$

After substitution for $\frac{d\xi}{dt}$ of $\xi \frac{\partial u}{\partial x}$, and for ρ from its definition, we obtain

$$\rho \xi \frac{du}{dt} + u \left[\xi \frac{d\rho}{dt} + \rho \xi \frac{\partial u}{\partial x} \right] = -\xi \frac{\partial}{\partial x} (\rho - \mu \frac{\partial u}{\partial x})$$

We note that the term in the brackets vanishes by virtue of (3a). Dividing then by $\rho \xi \neq 0$, we obtain

$$(3b) \quad \frac{du}{dt} = -\frac{1}{\rho} \frac{\partial}{\partial x} (\rho - \mu \frac{\partial u}{\partial x})$$

Again, we differentiate

$$(2c) \quad \frac{d}{dt} \left(\rho \xi \left[E + \frac{u^2}{2} \right] \right) = \xi \left[-\frac{\partial (p, u)}{\partial x} + \frac{\partial}{\partial x} \left(\lambda \frac{\partial T}{\partial x} \right) \right]$$

and substitute for ρ from its definition. This yields

$$\begin{aligned} \left[\frac{d\rho}{dt} \xi + \frac{d\xi}{dt} \rho \right] \left(E + \frac{u^2}{2} \right) + \rho \xi \left[\frac{dE}{dt} + u \frac{\partial u}{\partial t} \right] \\ = -\xi \frac{\partial}{\partial x} \left[u \left(\rho - \mu \frac{\partial u}{\partial x} \right) - \lambda \frac{\partial T}{\partial x} \right] \end{aligned}$$

We note once more that the term in the first brackets vanishes because of relation (3a). We substitute for $\frac{du}{dt}$ from (3b) and divide the resulting equation by ρ to obtain

$$\rho \left[\frac{dE}{dt} - \frac{u}{\rho} \frac{\partial}{\partial x} \left(p - \mu \frac{\partial u}{\partial x} \right) \right] = - \frac{\partial}{\partial x} \left[u \left(p - \mu \frac{\partial u}{\partial x} \right) - \lambda \frac{\partial I}{\partial x} \right]$$

We now divide through by ρ and carry the differentiation indicated on the right hand side one step further. This gives us

$$\frac{dE}{dt} - \frac{u}{\rho} \frac{\partial}{\partial x} \left(p - \mu \frac{\partial u}{\partial x} \right) = - \frac{u}{\rho} \frac{\partial}{\partial x} \left(p - \mu \frac{\partial u}{\partial x} \right) - \frac{1}{\rho} \left[\frac{\partial u}{\partial x} \left(p - \mu \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial x} \left(\lambda \frac{\partial I}{\partial x} \right) \right]$$

Hence,

$$\frac{dE}{dt} = - \frac{1}{\rho} \left[\left(p - \mu \frac{\partial u}{\partial x} \right) \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} \left(\lambda \frac{\partial I}{\partial x} \right) \right]$$

After substitution from (3a) for $\frac{\partial u}{\partial x}$, we finally obtain

$$(3c) \quad \frac{dE}{dt} = \left(p - \mu \frac{\partial u}{\partial x} \right) \frac{1}{\rho^2} \frac{d\rho}{dt} + \frac{1}{\rho} \frac{\partial}{\partial x} \left(\lambda \frac{\partial I}{\partial x} \right)$$

APPENDIX III

Transformation of Equation (3c)

$$(3c) \quad \frac{dE}{dt} = \left(p - \mu \frac{\partial u}{\partial x} \right) \frac{1}{\rho^2} \frac{d\rho}{dt} + \frac{1}{\rho} \frac{\partial}{\partial x} \left(\lambda \frac{\partial T}{\partial x} \right)$$

may be altered by introducing

$$T ds = dE + p dv = dE - p \frac{d\rho}{\rho^2}$$

From this,

$$dE = T ds + \frac{p}{\rho^2} d\rho$$

whence

$$\frac{dE}{dt} = T \frac{ds}{dt} + \frac{p}{\rho^2} \frac{d\rho}{dt}$$

When this is substituted into (3c), we obtain

$$T \frac{ds}{dt} + \frac{p}{\rho^2} \frac{d\rho}{dt} = \frac{p}{\rho^2} \frac{d\rho}{dt} - \frac{\mu}{\rho^2} \frac{\partial u}{\partial x} \frac{d\rho}{dt} + \frac{1}{\rho} \frac{\partial}{\partial x} \left(\lambda \frac{\partial T}{\partial x} \right)$$

After cancellation of like terms on both sides and replacement of $\frac{d\rho}{dt}$ by $-\rho \frac{\partial u}{\partial x}$ from (3a), we find

$$\rho T \frac{ds}{dt} = \mu \left(\frac{\partial u}{\partial x} \right)^2 + \frac{\partial}{\partial x} \left(\lambda \frac{\partial T}{\partial x} \right),$$

the transformed equation (3c).

APPENDIX IV

Conversion of $\frac{1}{\rho} \sqrt{\frac{dp}{d\rho}} + \frac{d}{d\rho} \sqrt{\frac{dp}{d\rho}}$ to a function of v, p .

We first note that $\rho = \frac{1}{v}$ by definition.
Hence, $dp = -\frac{1}{v^2} dv = -\rho^2 dv$, and $\frac{dp}{dv} = -\rho^2$.

Further,

$$\frac{dp}{dv} = \frac{dp}{d\rho} \frac{d\rho}{dv} = -\rho^2 \frac{dp}{d\rho},$$

making

$$\rho \sqrt{\frac{dp}{d\rho}} = \sqrt{-\frac{dp}{dv}}.$$

Also,

$$\begin{aligned} \frac{d}{dv} \sqrt{-\frac{dp}{dv}} &= \rho \frac{d}{d\rho} \sqrt{\frac{dp}{d\rho}} \frac{d\rho}{dv} + \sqrt{\frac{dp}{d\rho}} \frac{d\rho}{dv} \\ &= -\rho^3 \frac{d}{d\rho} \sqrt{\frac{dp}{d\rho}} - \rho^2 \sqrt{\frac{dp}{d\rho}} \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{\rho} \sqrt{\frac{dp}{d\rho}} + \frac{d}{d\rho} \sqrt{\frac{dp}{d\rho}} &= -v^3 \frac{d}{dv} \sqrt{-\frac{dp}{dv}} \\ &= \frac{v^3}{2 \sqrt{-\frac{dp}{dv}}} \frac{d^2 p}{dv^2} \end{aligned}$$

APPENDIX V

Solution of Differential Equations for Stationary Compression Shocks.

The three equations we are to solve are

$$(A) \quad u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0$$

$$(B) \quad \rho u \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} (p - \mu \frac{\partial u}{\partial x}) = 0$$

$$(C) \quad \rho u \frac{\partial E}{\partial x} = (p - \mu \frac{\partial u}{\partial x}) \frac{u}{\rho} \frac{\partial \rho}{\partial x} + \frac{\partial}{\partial x} (\lambda T)$$

The first of these may by inspection be written as

$$\frac{\partial}{\partial x} (\rho u) = 0,$$

whence $\rho u = \frac{u}{v} = M$; M being a constant of integration.

Equation (B) may be written more simply if we introduce M by writing $M \frac{\partial u}{\partial x}$ for $\rho u \frac{\partial u}{\partial x}$, and also noting that $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \frac{dv}{dx} = M \frac{dv}{dx}$. Integration

after performance of these substitutions, yields

$Mu + p - \mu M \frac{dv}{dx} = J$; where J is another constant of integration. Using the solution of (A), this becomes

$$M^2 v + p - J = \mu M \frac{dv}{dx}$$

In order to solve (C), we note

$$\frac{u}{\rho} \frac{\partial \rho}{\partial x} = uv \frac{\partial \rho}{\partial v} \frac{dv}{dx} = - \frac{uv}{v^2} \frac{dv}{dx} = -M \frac{dv}{dx}$$

and substitute $\rho u = M$ and $p - \mu \frac{\partial u}{\partial x} = J - Mu$ from foregoing solutions to obtain

$$M \frac{\partial E}{\partial x} = -(J - Mu) M \frac{dv}{dx} + \frac{\partial}{\partial x} (\lambda \frac{\partial T}{\partial x}), \text{ or}$$

$$\frac{\partial E}{\partial x} = -J \frac{dv}{dx} + Mu \frac{dv}{dx} + \frac{1}{M} \frac{\partial}{\partial x} (\lambda \frac{\partial T}{\partial x})$$

After substituting $M^2 v$ for Mu , we may integrate this to obtain

$$E + Jv - \frac{1}{2} M^2 v^2 - F = \frac{\lambda}{M} \frac{\partial T}{\partial x}.$$

F here is another constant of integration.

APPENDIX VI

check of Equations (19) using definitions (18)

We note that

$$(15c) \quad E_2 - E_1 = \frac{1}{2} (p_1 + p_2) (v_1 - v_2)$$

$$(17b) \quad E_2 - E_1 = \bar{c}_v (T_2 - T_1)$$

Hence,

$$2 \bar{c}_v = \frac{(p_1 + p_2) (v_1 - v_2)}{T_2 - T_1}$$

But, (17a) states $p v = RT$, hence

$$\frac{2 \bar{c}_v}{R} = \frac{(p_1 + p_2) (v_1 - v_2)}{p_2 v_2 - p_1 v_1}$$

By (18), $\xi_1 = \frac{2 \bar{c}_v}{R} + 1$, hence

$$\begin{aligned} \xi_1 &= \frac{p_2 v_1 + p_2 v_1 - p_1 v_2 - p_2 v_2 + (p_2 v_2 - p_1 v_1)}{p_2 v_2 - p_1 v_1} \\ &= \frac{p_2 v_1 - p_1 v_2}{p_2 v_2 - p_1 v_1} \end{aligned}$$

And, since (18) also defines $\pi = p_2/p_1$

$$\begin{aligned} \frac{\pi \xi_1 + 1}{\xi_1 + \pi} &= \frac{\frac{p_2}{p_1} \cdot \frac{p_2 v_1 - p_1 v_2}{p_2 v_2 - p_1 v_1} + 1}{\frac{p_2 v_1 - p_1 v_2}{p_2 v_2 - p_1 v_1} + \frac{p_2}{p_1}} = \frac{p_2^2 v_1 - p_1 p_2 v_2 + p_1 p_2 v_2 - p_1^2 v_1}{p_1 p_2 v_1 - p_1^2 v_2 + p_2^2 v_2 - p_1 p_2 v_1} \\ &= \frac{v_1 (p_2^2 - p_1^2)}{v_2 (p_2^2 - p_1^2)} = \frac{v_1}{v_2} \end{aligned}$$

We have hence checked (19b).

To check (19c), we use (17a) to write
 $p_2 v_2 = RT_2$ and $p_1 v_1 = RT_1$

Hence,

$$\frac{p_2 v_2}{p_1 v_1} = \frac{T_2}{T_1} \quad \text{or}$$

$$\frac{T_2}{T_1} = \pi \frac{v_2}{v_1} \quad \text{from the definition of } \pi.$$

(19a) may then be obtained directly by substituting from (19b) into the above. Hence (19a) is verified.

$$\begin{aligned}
p_1 v_1 \frac{\pi \xi_1 + 1}{\xi_1 - 1} &= p_1 v_1 \frac{\frac{p_2^2 v_1 - p_1^2 v_1}{(p_2 v_2 - p_1 v_1) p_1}}{\frac{p_2 v_1 - p_1 v_2 - p_2 v_2 + p_1 v_1}{(p_2 v_2 - p_1 v_1)}} \\
&= \frac{p_1 v_1^2 (p_2^2 - p_1^2)}{p_1 (p_2 + p_1) (v_1 - v_2)} = v_1^2 \frac{p_2 - p_1}{v_1 - v_2} = D^2 \text{ from (16)}
\end{aligned}$$

Thus, we have checked (19c).

Substituting into (19d) from previous work, we obtain

$$\begin{aligned}
p_1 v_1 (\xi_1 - 1) \frac{(\pi - 1)^2}{\pi \xi_1 + 1} &= p_1 v_1 \left(\frac{p_2 v_1 - p_1 v_2 - p_2 v_2 + p_1 v_1}{p_2 v_2 - p_1 v_1} \right) \frac{\left(\frac{p_2 - p_1}{p_1} \right)^2}{\frac{v_1 (p_2^2 - p_1^2)}{p_1 (p_2 v_2 - p_1 v_1)}} \\
&= \frac{(p_2 + p_1) (v_1 - v_2) \cancel{p_1 v_1}}{(p_2 v_2 - p_1 v_1)} \cdot \frac{(p_2 - p_1)^2}{\cancel{p_1^2}} \cdot \frac{\cancel{p_1} (p_2 v_2 - p_1 v_1)}{\cancel{p_1} (p_2^2 - p_1^2)} \\
&= \frac{(v_1 - v_2) (p_2 + p_1) (p_2 - p_1)^2}{(p_2 - p_1) (p_2 + p_1)} \\
&= W^2 \text{ from (16)}
\end{aligned}$$

Hence, (19d) has been verified also.

APPENDIX VII

Transformation of Hugoniot Conditions (14)

We have shown that the relations between the properties of a fluid on two sides of a stationary shock front are

$$(14a) \quad \frac{u_1}{v_1} = \frac{u_2}{v_2}$$

$$(14b) \quad \frac{u_1^2}{v_1} + p_1 = \frac{u_2^2}{v_2} + p_2$$

$$(14c) \quad E_1 + \frac{u_1^2}{2} + p_1 v_1 = E_2 + \frac{u_2^2}{2} + p_2 v_2$$

By manipulation of the above equations we have obtained

$$(15a) \quad u_1^2 = v_1^2 \cdot \frac{p_2 - p_1}{v_1 - v_2}$$

$$(15b) \quad u_2^2 = v_2^2 \cdot \frac{p_2 - p_1}{v_1 - v_2}$$

If we now wish to refer the motion to a reference frame that is stationary rather than to one moving with the shock wave, we write

$$(U - u) = u_2, \quad (U - u_0) = u_1$$

where U denotes the velocity of the shock wave, u_0 is the particle velocity of the undisturbed fluid ahead of the shock, and u that of the fluid behind the shock. We wish to consider the fluid initially at rest, i.e. $u_0 = 0$, so that $U = u_1$. Substituting these, and replacing v by $\frac{1}{\rho}$, the above equations become (after a change of subscripts to agree with the new notation):

$$(14a-1) \quad \rho_0 U = (U - u) \rho$$

$$(14b-1) \quad U^2 \rho_0 + p_0 = (U - u)^2 \rho + p$$

$$(14c-1) \quad E_0 + \frac{1}{2} U^2 + p_0 v_0 = E + \frac{1}{2} (U - u)^2 + p v$$

$$(15a-1) \quad U^2 = v_0^2 \cdot \frac{p - p_0}{v_0 - v}$$

$$(15b-1) \quad (U - u)^2 = v^2 \cdot \frac{p - p_0}{v_0 - v}$$

Letting $P = p - p_0$, we obtain from (15a-1) that

$$P = U^2 \rho_0^2 (v_0 - v) = \rho_0 U [\rho_0 U (v_0 - v)]$$

But, from (14a-1)

$$(\rho_0 - \rho) U = -u \rho,$$

making

$$u = \frac{U}{\rho} (\rho - \rho_0) = U \left(1 - \frac{\rho_0}{\rho}\right) = U \rho_0 \left(\frac{1}{\rho_0} - \frac{1}{\rho}\right) = U \rho_0 (v_0 - v)$$

Hence,

$$P = \rho_0 u U.$$

Equ. (14c-1) may be written

$$E - E_0 + p v - p_0 v_0 = \frac{1}{2} U^2 - \frac{1}{2} (U - u)^2$$

or, since $H = E + p v$,

$$H - H_0 = \Delta H = \frac{1}{2} [U^2 - (U - u)^2]$$

Substituting from (15a-1) and (15b-1)

$$\begin{aligned} \Delta H &= \frac{1}{2} \left[v_0^2 \frac{P}{v_0 - v} - v^2 \frac{P}{v_0 - v} \right] \\ &= \frac{P}{2} \frac{v_0^2 - v^2}{v_0 - v} \\ &= \frac{P}{2} (v_0 + v) = \frac{P}{2} \left(\frac{1}{\rho_0} + \frac{1}{\rho} \right) \end{aligned}$$

We have hence derived the form of the Hugoniot conditions given by Brinkley and Kirkwood:

$$P = \rho_0 u U$$

$$\rho (U - u) = \rho_0 U$$

$$\Delta H = \frac{P}{2} \left(\frac{1}{\rho_0} + \frac{1}{\rho} \right)$$

from Becker's equations.

APPENDIX VIII

Application of $\frac{d}{dR} = \frac{\partial}{\partial R} + \frac{1}{U} \frac{\partial}{\partial t}$ to $P = \rho_0 u U$

Direct application of the operator to both sides of the equation yields, after division through by ρ_0 :

$$-\frac{1}{\rho_0} \frac{\partial P}{\partial R} - \frac{1}{\rho_0 U} \frac{\partial P}{\partial t} + u \left(\frac{\partial U}{\partial R} + \frac{1}{U} \frac{\partial U}{\partial t} \right) + U \left(\frac{\partial u}{\partial R} + \frac{1}{U} \frac{\partial u}{\partial t} \right) = 0$$

Differentiating $\frac{P}{\rho_0} = u U$ with respect to P , we find

$$\frac{1}{\rho_0} = u \frac{dU}{dP} + U \frac{du}{dP}$$

whence $U \frac{du}{dP} = \frac{1}{\rho_0} - u \frac{dU}{dP}$

If we define

$$g = \rho_0 U \frac{du}{dP},$$

then

$$g = \rho_0 \left(\frac{1}{\rho_0} - u \frac{dU}{dP} \right) = 1 - \rho_0 u \frac{dU}{dP} = 1 - \frac{P}{U} \frac{dU}{dP}$$

Further,

$$-\frac{g}{\rho_0} \frac{\partial P}{\partial R} = \left(u \frac{dU}{dP} - \frac{1}{\rho_0} \right) \frac{\partial P}{\partial R} = u \frac{\partial U}{\partial R} - \frac{1}{\rho_0} \frac{\partial P}{\partial R}$$

and

$$\begin{aligned} -\frac{g}{\rho_0 U} \frac{\partial P}{\partial t} &= \left(\frac{P}{\rho_0 U^2} \frac{dU}{dP} - \frac{1}{\rho_0 U} \right) \frac{\partial P}{\partial t} = \frac{P}{\rho_0 U^2} \frac{dU}{dt} - \frac{1}{\rho_0 U} \frac{\partial P}{\partial t} \\ &= \frac{u}{U} \frac{\partial U}{\partial t} - \frac{1}{\rho_0 U} \frac{\partial P}{\partial t} \end{aligned}$$

Hence, the expression obtained by application of the operator may be written more simply

$$\frac{\partial U}{\partial t} + U \frac{\partial u}{\partial R} - \frac{g}{\rho_0} \frac{\partial P}{\partial R} - \frac{g}{\rho_0 U} \frac{\partial P}{\partial t} = 0$$

APPENDIX IX

$$\underline{\text{check of } D(R) = R^2 P u \mu V}$$

By definition of T , $T = \frac{1}{\mu} (t - t_0(R))$

Then $dT = \frac{1}{\mu} dt$ since

$$\frac{1}{\mu} = - \frac{\partial}{\partial t} (\log P' u' r^2)_{t=t_0(R)} \text{ is independent of } t.$$

Also, when $t = t_0(R)$, $T = 0$

and when $t = \infty$, $T = \infty$

Hence,

$$V = \int_0^\infty f(R, T) dT = \int_{t_0(R)}^\infty f(R, T) \frac{dt}{\mu}$$

$$\text{But } f(R, T) = \frac{r^2 P' u'}{R^2 P u}$$

Then,

$$\begin{aligned} R^2 P u \mu V &= R^2 P u \mu \int_{t_0(R)}^\infty \frac{r^2 P' u'}{R^2 P u} \frac{dt}{\mu} \\ &= \int_{t_0(R)}^\infty r^2 P' u' dt \end{aligned}$$

since R, P, u, μ are independent of time t
(since we are dealing with a fixed R).

APPENDIX X

Manipulations in Derivation of Equations (35)

Differentiation of

$$(31) \quad D(R) = \int_R^{\infty} h \rho_0 r^2 dr$$

yields

$$\frac{dD}{dR} = -\rho_0 R^2 h$$

Letting $L(P) = \rho_0 h$, we obtain at once the first of equations (35):

$$\frac{dD}{dR} = -R^2 L(P)$$

In order to obtain an expression for $\frac{dP}{dR}$, we wish to use

$$(28) \quad \frac{dP}{dR} = \frac{\partial P}{\partial R} + \frac{1}{U} \frac{\partial P}{\partial t}$$

The partial derivatives of P are obtained by simultaneous solution of equations (25c), (25d), (27), and (34). We re-write these as follows.

$$(34') \quad \frac{1}{P} \frac{\partial P}{\partial t} + \frac{1}{u} \frac{\partial u}{\partial t} = - \left(\frac{R^2 P u u}{D} + \frac{2u}{R} \right)$$

$$(25d') \quad \frac{1}{\rho_0^2} \frac{\partial P}{\partial t} + \frac{P}{\rho_0} \frac{\partial u}{\partial R} = - \frac{2u}{R}$$

$$(25c') \quad \frac{1}{\rho_0} \frac{\partial P}{\partial R} + \frac{\partial u}{\partial t} = 0$$

$$(27') \quad -\frac{g}{\rho_0 U} \frac{\partial P}{\partial t} - \frac{g}{\rho_0} \frac{\partial P}{\partial R} + \frac{\partial u}{\partial t} + U \frac{\partial u}{\partial R} = 0$$

We now use Kramer's rule for solution of these equations by determinants. Defining Q as

$$Q = \begin{vmatrix} \frac{1}{P} & 0 & \frac{1}{u} & 0 \\ \frac{1}{\rho_0^2} & 0 & 0 & \frac{P}{\rho_0} \\ 0 & \frac{1}{\rho_0} & 1 & 0 \\ -\frac{g}{\rho_0 U} & -\frac{g}{\rho_0} & 1 & U \end{vmatrix}$$

we evaluate it :

$$\begin{aligned}
 Q &= -\frac{1}{\rho c^2} \begin{vmatrix} 0 & \frac{1}{\rho_0} & 0 \\ \frac{1}{\rho_0} & 1 & 0 \\ -\frac{g}{\rho_0} & 1 & U \end{vmatrix} + \frac{p}{\rho_0} \begin{vmatrix} \frac{1}{p} & 0 & \frac{1}{u} \\ 0 & \frac{1}{\rho} & 1 \\ -\frac{g}{\rho u} & -\frac{g}{\rho_0} & 1 \end{vmatrix} \\
 &= -\frac{U}{\rho c^2} \left(-\frac{1}{\rho_0} \right) + \frac{p}{\rho_0} \left[\frac{1}{p} \left(\frac{1}{\rho_0} + \frac{g}{\rho_0} \right) - \frac{g}{\rho_0 U} \left(-\frac{1}{\rho_0 U} \right) \right] \\
 &= \frac{U}{\rho \rho_0 u c^2} + \frac{p}{\rho_0} \left[\frac{1}{p \rho_0} + \frac{g}{p \rho_0} + \frac{g}{\rho_0^2 U u} \right] \\
 &= \frac{U}{\rho \rho_0 u c^2} + \frac{(1+g)p}{\rho_0^2} + \frac{p g}{\rho_0^2 U u}
 \end{aligned}$$

Further,

$$\begin{aligned}
 Q \frac{\partial p}{\partial R} &= \begin{vmatrix} \frac{1}{p} & -\left(\frac{R^2 p u v}{D} + \frac{2u}{R} \right) & \frac{1}{u} & 0 \\ \frac{1}{\rho c^2} & -\frac{2u}{R} & 0 & \frac{p}{\rho_0} \\ 0 & 0 & 1 & 0 \\ -\frac{g}{\rho_0 U} & 0 & 1 & U \end{vmatrix} \\
 &= \begin{vmatrix} \frac{1}{p} & -\left(\frac{R^2 p u v}{D} + \frac{2u}{R} \right) & 0 & 0 \\ \frac{1}{\rho c^2} & -\frac{2u}{R} & \frac{p}{\rho_0} & 0 \\ -\frac{g}{\rho_0 U} & 0 & U & 1 \end{vmatrix} \\
 &= \frac{1}{p} \left(-\frac{2u}{R} U \right) + \left(\frac{R^2 p u v}{D} + \frac{2u}{R} \right) \left(\frac{U}{\rho c^2} + \frac{g}{\rho_0^2} \right)
 \end{aligned}$$

and

$$\begin{aligned}
 Q \frac{\partial p}{\partial t} &= \begin{vmatrix} -\left(\frac{R^2 p u v}{D} + \frac{2u}{R} \right) & 0 & \frac{1}{u} & 0 \\ -\frac{2u}{R} & 0 & 0 & \frac{p}{\rho_0} \\ 0 & \frac{1}{\rho_0} & 1 & 0 \\ 0 & -\frac{g}{\rho_0} & 1 & U \end{vmatrix} \\
 &= -\left(\frac{R^2 p u v}{D} + \frac{2u}{R} \right) \begin{vmatrix} 0 & 0 & \frac{p}{\rho_0} \\ \frac{1}{\rho_0} & 1 & 0 \\ -\frac{g}{\rho_0} & 1 & U \end{vmatrix} + \frac{2u}{R} \begin{vmatrix} 0 & \frac{1}{u} & 0 \\ \frac{1}{\rho_0} & 1 & 0 \\ -\frac{g}{\rho_0} & 1 & U \end{vmatrix} \\
 &= -\left(\frac{R^2 p u v}{D} + \frac{2u}{R} \right) \frac{p}{\rho_0} \left(\frac{1}{\rho_0} + \frac{g}{\rho_0} \right) + \frac{2u}{R} \left(-\frac{1}{u} \right) \left(\frac{U}{\rho_0} \right) \\
 &= -\left(\frac{R^2 p u v}{D} + \frac{2u}{R} \right) \frac{p}{\rho_0^2} (1+g) - \frac{2U}{R \rho_0}
 \end{aligned}$$

Substituting the foregoing into (28), we find

$$\frac{dP}{dR} = \frac{1}{Q} \left[-\frac{2uU}{PR} + \left(\frac{R^2 Pu v}{D} + \frac{2u}{R} \right) \left(\frac{U}{\rho c^2} + \frac{gP}{U \rho_0^2} \right) - \frac{(1+g)}{U} \frac{\rho}{\rho_0^2} \left(\frac{R^2 Pu v}{D} + \frac{2u}{R} \right) - \frac{2}{R \rho_0} \right]$$

which we may write as

$$\frac{dP}{dR} = A v + B$$

where

$$A = \frac{1}{Q} \left[\frac{R^2 Pu}{D} \left(\frac{U}{\rho c^2} + \frac{g^2}{U \rho_0^2} - \frac{(1+g)}{U} \frac{\rho}{\rho_0^2} \right) \right]$$

$$B = \frac{1}{Q} \left[-\frac{2uU}{PR} + \frac{2u}{R} \left(\frac{U}{\rho c^2} + \frac{gP}{U \rho_0^2} \right) - \frac{(1+g)}{U} \frac{\rho}{\rho_0^2} \frac{2u}{R} - \frac{2}{R \rho_0} \right]$$

From the first two of equations (4.6),

$$u = \frac{P}{\rho_0 U} = \left(1 - \frac{\beta}{\rho} \right) U,$$

$$P = \left(1 - \frac{\rho_0}{\rho} \right) \rho_0 U^2$$

Using these relations, we find

$$A = \frac{1}{Q} \left[\frac{R^2 P}{D} \cdot \frac{P}{\rho_0 U} \left(\frac{U}{\rho c^2} - \frac{\rho}{U \rho_0^2} \right) \right]$$

$$B = \frac{1}{Q} \left[-\frac{2U}{RR} \frac{P}{\rho_0 U} + \frac{2}{R} \frac{P}{\rho_0 U} \left(\frac{U}{\rho c^2} + \frac{g^2}{U \rho_0^2} \right) - \frac{2(1+g)\rho}{UR \rho_0^2} \frac{P}{\rho_0 U} - \frac{2}{R \rho_0} \right]$$

Simplifying A and B further,

$$\begin{aligned} A &= \frac{\frac{R^2 P^2}{D \rho_0 U} \left(\frac{U}{\rho c^2} - \frac{\rho}{U \rho_0^2} \right)}{\frac{1}{\rho_0} \left[\frac{U^2 \rho_0}{\rho c^2 P} + \frac{(1+g)\rho}{P \rho_0} + \frac{\rho g \rho_0}{\rho_0^2 P} \right]} = \frac{\frac{R^2 P^2}{D U^2} \left[\frac{U^2}{\rho c^2} - \frac{\rho}{\rho_0^2} \right]}{\left[\frac{U^2}{c^2} \frac{\rho_0}{\rho} \frac{1}{P} + \frac{(1+g)\rho}{P \rho_0} + \frac{\rho g}{P \rho_0} \right]} \\ &= \frac{\frac{R^2 P^3}{D U^2 \rho_0} \left[\frac{U^2}{c^2} \frac{\rho_0^2}{\rho^2} - 1 \right]}{\left[\frac{U^2}{c^2} \frac{\rho_0^2}{\rho^2} + 1 + 2g \right]} = \frac{-R^2 P^3 G}{D \rho_0 U^2 [2(1+g) - G]} \end{aligned}$$

$$\text{where } G = 1 - \left(\frac{\rho_0 U}{\rho c} \right)^2$$

$$\begin{aligned}
B &= \frac{-\frac{4}{R\rho_0} + \frac{2P}{R\rho_0\rho c^2} + \frac{2gP\rho}{R\rho_0^3 U^2} - \frac{2(1+g)P\rho}{R\rho_0^3 U^2}}{\frac{1}{\rho_0} \left[\frac{U^2\rho_0}{\rho c^2 P} + \frac{(1+g)\rho}{P\rho_0} + \frac{g\rho\rho_0}{\rho_0^3 P} \right]} \\
&= \frac{\frac{P\rho_0}{P} \left[-\frac{4}{R} + \frac{2P}{R\rho c^2} - \frac{2P\rho}{R\rho_0^2 U^2} \right]}{\frac{U^2\rho_0}{c^2\rho P} \cdot \frac{P\rho_0}{P} + \frac{(1+g)\rho}{P\rho_0} \cdot \frac{P\rho_0}{P}} = \frac{\frac{P}{R} \left[-4\frac{\rho_0}{P} + 2\left(\frac{2P\rho_0}{\rho c^2 P} - \frac{2P\rho}{\rho_0^2 U^2 P}\right) \right]}{\frac{U^2}{c^2} \frac{\rho_0^2}{P^2} + 1 + 2g} \\
&= \frac{P}{R[2(1+g) - G]} \left[-4\frac{\rho_0}{P} - 2\left(1 - \frac{\rho_0}{P}\right)\rho_0 U^2 \left(1 - \frac{\rho_0}{\rho c^2}\right) \right] \\
&= -\frac{P}{R} \frac{4\frac{\rho_0}{P} + 2\left(1 - \frac{\rho_0}{P}\right)G}{2(1+g) - G}
\end{aligned}$$

If we now define

$$\begin{aligned}
M(P) &= \frac{1}{\rho_0 U^2} \frac{G}{2(1+g) - G} \\
N(P) &= \frac{4\frac{\rho_0}{P} + 2\left(1 - \frac{\rho_0}{P}\right)G}{2(1+g) - G}
\end{aligned}$$

we may write

$$\frac{dP}{dR} = -2 \frac{R^2 P^3}{D} M(P) - \frac{P}{R} N(P)$$

APPENDIX XI

Illustration of conversion of L, M, N, h, to functions of P only, for an ideal gas.

Before equations (35) can be integrated numerically, we must express all values as functions of R and P only, which necessitates expression of h, L, M, N, as functions of P only. We illustrate this procedure for an ideal gas, making use of its equation of state

$$(17a) \quad p v = R T, \quad \text{and of}$$

$$(17b) \quad E - E_0 = c_v (T - T_0)$$

But, for an ideal gas, $c_v = \frac{R}{k-1}$ (Obert, "Thermodynamics", p 187, equ. 7-12), making (17b)

$$E - E_0 = \frac{R}{k-1} (T - T_0) = \frac{p v - p_0 v_0}{k-1}$$

Equating the Hugoniot Equation

$$(15c) \quad E - E_0 = \frac{1}{2} (p + p_0) (v_0 - v)$$

to the above, we obtain

$$\frac{1}{k-1} (p v - p_0 v_0) = \frac{1}{2} (v_0 - v)$$

which we solve for v, to obtain

$$v = v(p) = v_0 \frac{p + (1 + \frac{2}{k-1}) p_0}{(1 + \frac{2}{k-1}) p + p_0}$$

To change $v(p)$ to $v(P)$, we recall the definition $P = p - p_0$, or $p = P + p_0$. With this substitution, we find

$$v(P) = v_0 \frac{P + \frac{2k}{k-1} p_0}{\frac{k+1}{k-1} P + \frac{2k}{k-1} p_0}$$

In order to evaluate $h(P)$, we refer to the third of equations (20):

$$h - h_0 = \frac{1}{2} P \left(\frac{1}{p_0} + \frac{1}{p} \right) = \frac{P}{2} (v_0 + v)$$

Substituting for $v = v(P)$, we find

$$h(P) = h_0 + \frac{P}{2} v_0 \left[1 + \frac{P + \frac{2K}{K-1} P_0}{\frac{K+1}{K-1} P + \frac{2K}{K-1} P_0} \right]$$

In order to express $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$ as a function of P only, we refer to the definition of entropy

$$T ds = dE + p dv$$

We note from our previous discussion that $dE = c_v dT = \frac{R}{K-1} dT = \frac{1}{K-1} d(pv) = \frac{1}{K-1} (p dv + v dp)$.

Then,

$$T ds = \frac{1}{K-1} (p dv + v dp) + p dv$$

Letting $ds = 0$, we find

$$\frac{1}{K-1} v dp = -p dv - \frac{1}{K-1} p dv = -p \left(1 + \frac{1}{K-1}\right) dv$$

Hence,

$$\left(\frac{\partial p}{\partial v}\right)_s = \frac{-p \left(\frac{K}{K-1}\right) dv}{\frac{v}{K-1}} = -\frac{p}{v} K$$

But $v = \frac{1}{\rho}$, and $\frac{dv}{d\rho} = -\frac{1}{\rho^2} = -v^2$,

making

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \left(\frac{\partial p}{\partial v}\right)_s \frac{dv}{d\rho} = -\frac{p}{v} K (-v^2) = K p v$$

Here we may substitute $p = P + p_0$ and $v = v(P)$ once more, but we shall find it more expedient to proceed along different lines since we are interested in determining $G(P)$, where

$$G = 1 - \left(\frac{p_0 v}{p c}\right)^2$$

using

$$(16) \quad U = v \sqrt{\frac{P - P_0}{v_0 - v}},$$

$$U^2 = v^2 \frac{(P - P_0)}{(v_0 - v)}, \text{ and } G \text{ becomes}$$

$$G = 1 - \frac{v^2}{v_0^2} \cdot \frac{v^2 \left(\frac{P - P_0}{v - v_0}\right)}{K p v} = 1 - \frac{v^3}{v_0^2} \frac{(P - P_0)}{K p (v_0 - v)}$$

Replacement of p by $P + p_0$, and substitution of $v(P)$ gives us

$$G(P) = 1 - \left[\frac{P + \frac{2K}{K-1} P_0}{\frac{K+1}{K-1} P + \frac{2K}{K-1} P_0} \right]^3 \frac{P}{K(P+P_0) \left(1 - \frac{P + \frac{2K}{K-1} P_0}{\frac{K+1}{K-1} P + \frac{2K}{K-1} P_0} \right)}$$

To evaluate $g(P)$, where $g = 1 - \frac{P}{U} \frac{dU}{dP}$ we return to

$$U^2 = v^2 \frac{(P-P_0)}{(v_0-v)},$$

substitute $p = P + p_0$ and $v(P)$ to obtain

$$U^2 = v_0^2 \left[\frac{P + \frac{2K}{K-1} P_0}{\frac{K+1}{K-1} P + \frac{2K}{K-1} P_0} \right]^2 \cdot \frac{P}{1 - \left[\frac{P + \frac{2K}{K-1} P_0}{\frac{K+1}{K-1} P + \frac{2K}{K-1} P_0} \right]}$$

Differentiating,

$$\frac{2U dU}{v_0 dP} = \left[\frac{P + \frac{2K}{K-1} P_0}{\frac{K+1}{K-1} P + \frac{2K}{K-1} P_0} \right]^2 \cdot \frac{\left[1 - \frac{P + \frac{2K}{K-1} P_0}{\frac{K+1}{K-1} P + \frac{2K}{K-1} P_0} \right] + P \cdot \frac{\frac{K+1}{K-1} P + \frac{2K}{K-1} P_0 - \frac{K+1}{K-1} (P + \frac{2K}{K-1} P_0)}{(\frac{K+1}{K-1} P + \frac{2K}{K-1} P_0)^2}}{\left[1 - \frac{P + \frac{2K}{K-1} P_0}{\frac{K+1}{K-1} P + \frac{2K}{K-1} P_0} \right]^2}$$

and

$$g = 1 - \frac{P v_0}{2} \cdot \frac{1}{P v_0} \left\{ 1 + P \cdot \frac{\left[\frac{K+1}{K-1} P + \frac{2K}{K-1} P_0 - \frac{K+1}{K-1} P - \frac{2K(K+1)}{(K-1)^2} P_0 \right] \left[\frac{2}{K-1} P \right]}{\left(\frac{K+1}{K-1} P + \frac{2K}{K-1} P_0 \right)^3} \right\}$$

$$= \frac{1}{2} \left[1 + \frac{2P^2 (-2K-1) P_0}{\left((K+1)P + 2K P_0 \right)^3} \right]$$

or

$$g = \frac{1}{2} - \frac{(2K+1) P_0 P^2}{\left[(K+1)P + 2K P_0 \right]^3}$$

Substitution of the above expressions into L, M, N, makes these too cumbersome for use. For numerical integration of (35), $G(P)$, $g(P)$, $v(P)$, $U^2(P)$ would be evaluated separately by the expressions derived here, and then the numerical values substituted into the original definitions of L, M, and N. (page 30)

x - t PLANE FOR BECKER'S PISTON

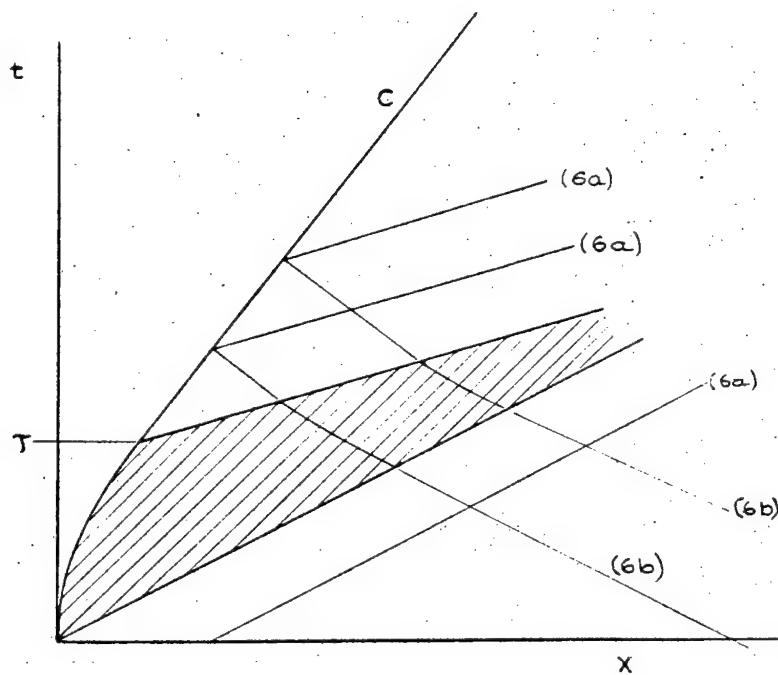


FIG. 1.

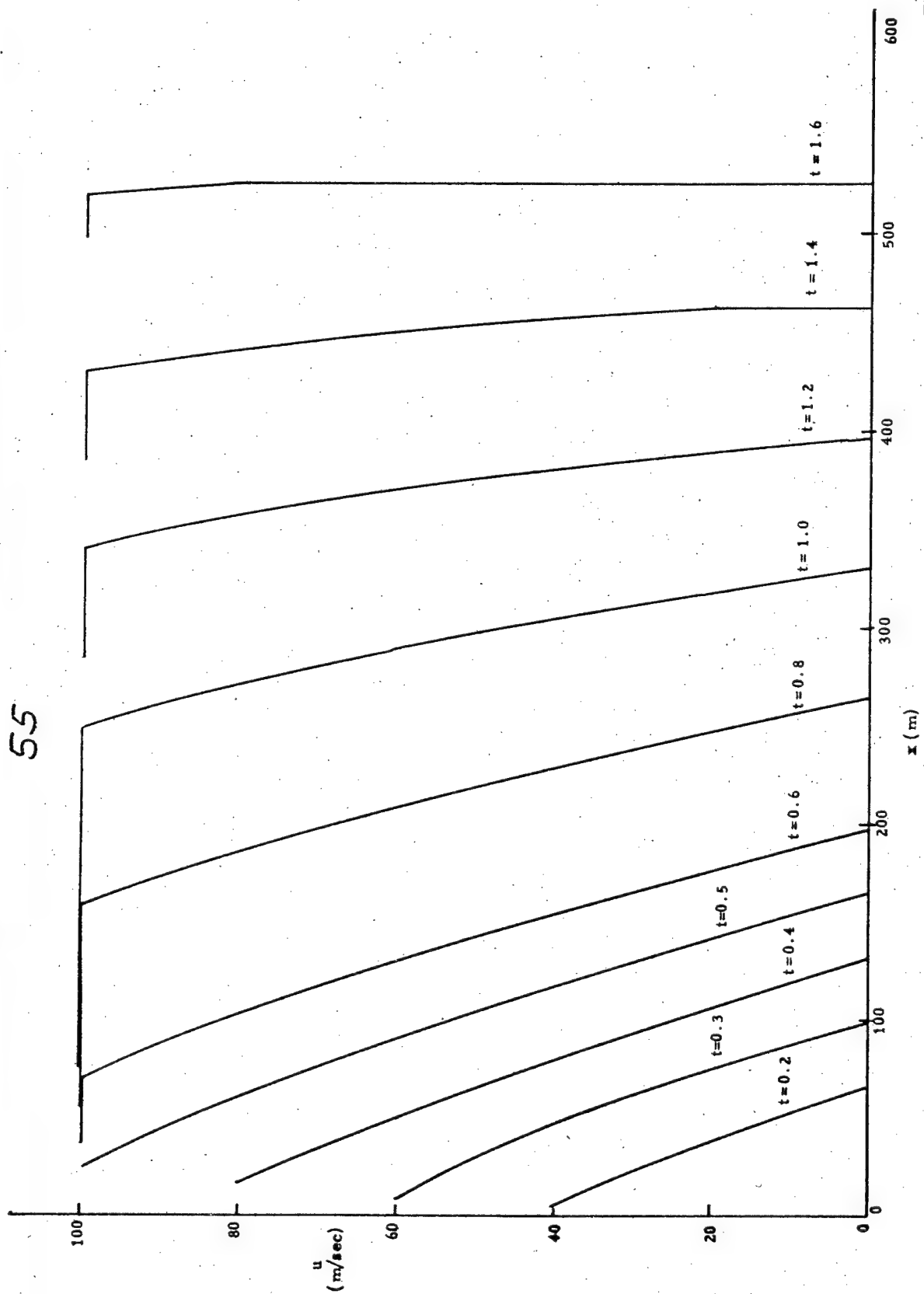


Fig. 2

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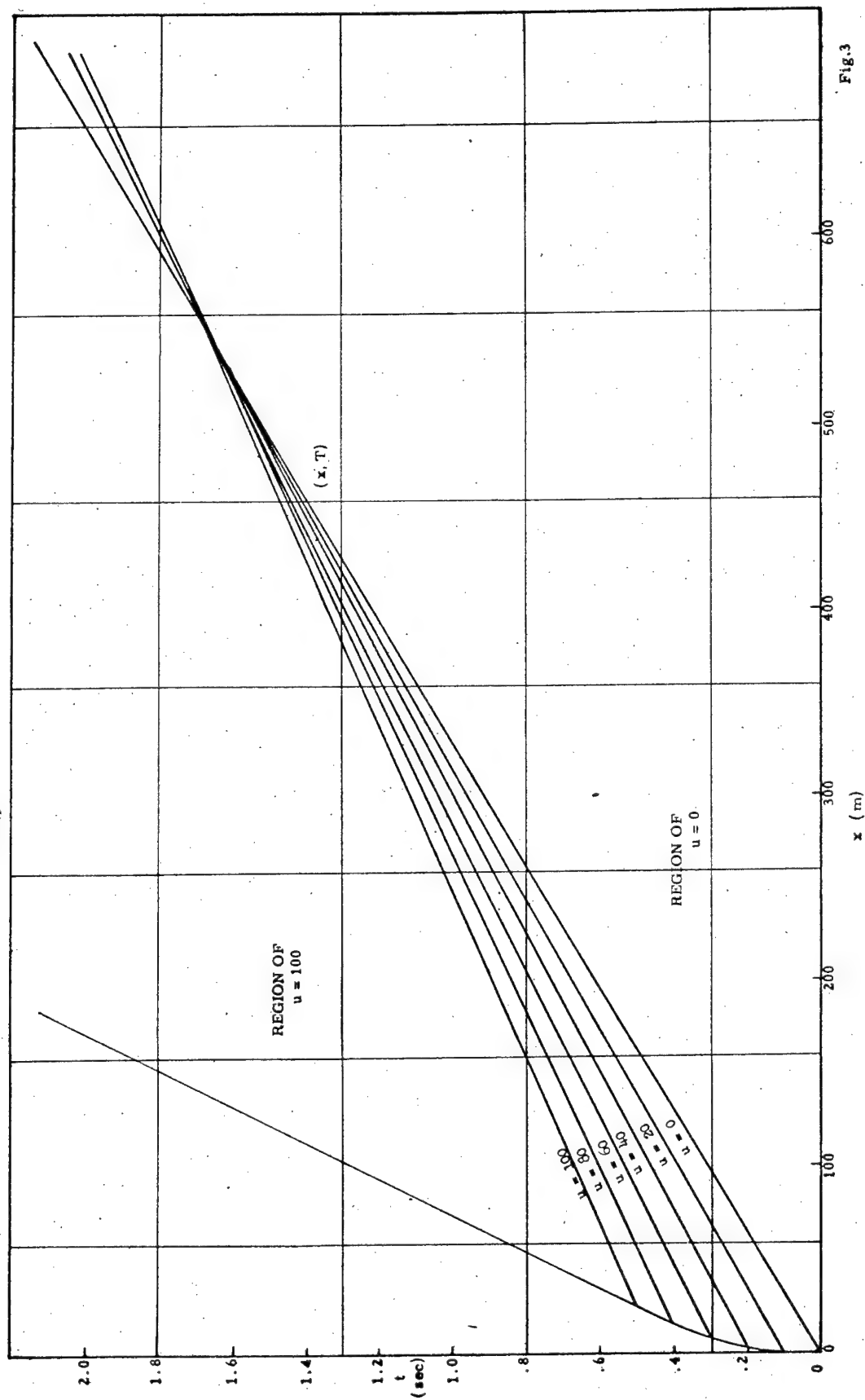


Fig. 3

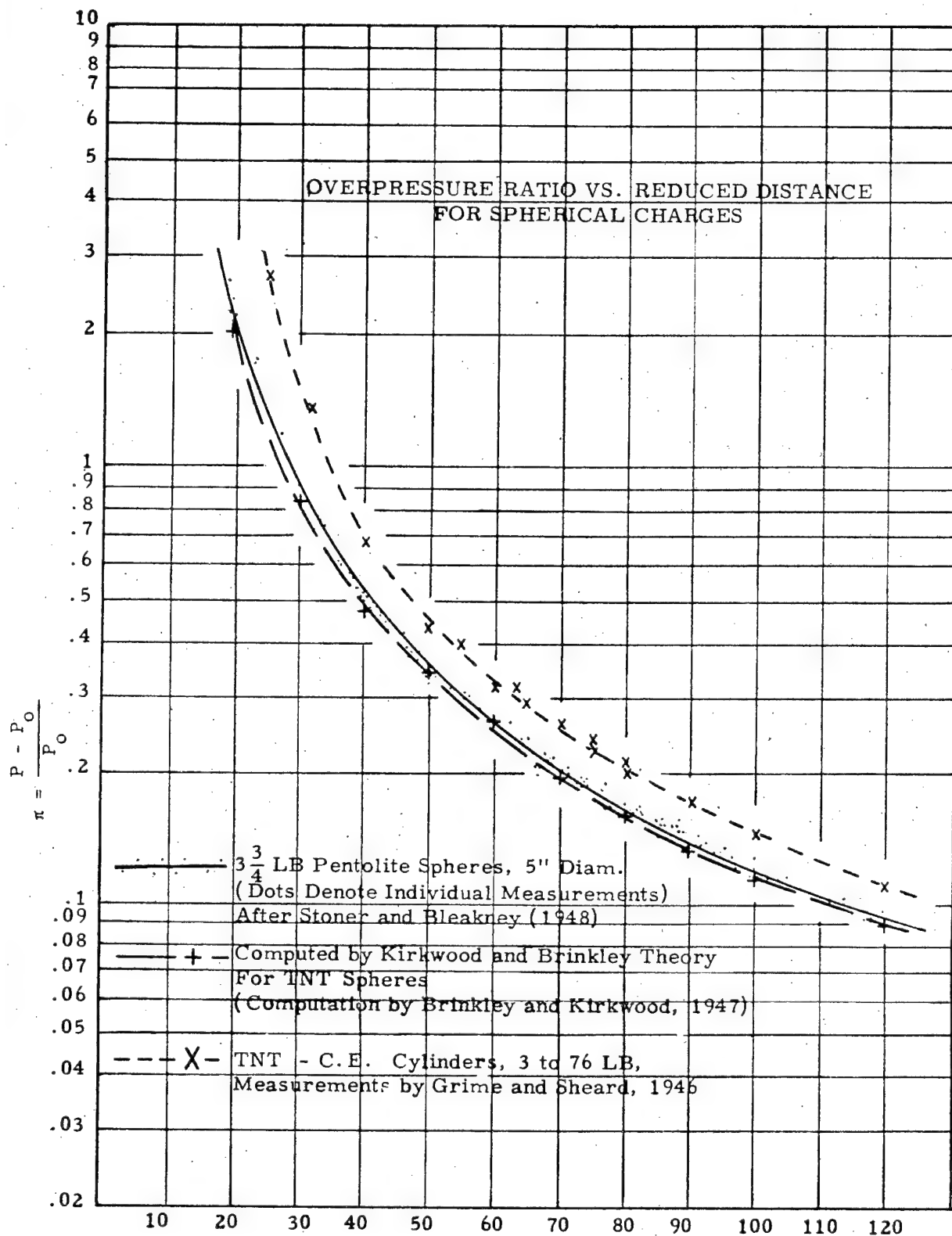


Fig. 4

GLOSSARY OF HYDRODYNAMIC EQUATIONS

Basic Approaches

Hydrodynamic relations may be expressed in two different forms, one due to Lagrange, the other due to Euler.

1. Lagrange's form describes fluid motion in terms of the paths of individual particles, i.e. it expresses the coordinates x, y, z , of particles as functions of time and three other parameters (usually taken as the initial conditions). Lagrange's equations are usually too cumbersome, but are advantageous for describing one-dimensional motion.
2. In Euler's form, an observer is considered at each point (x, y, z) , and the happenings at these points are observed in the course of time t . The fluid motion is described by giving as functions of x, y, z, t , the velocity components u, v, w , of the particle that is at x, y, z , at time t .
3. The relation between the two systems is expressed by the equations
$$\begin{aligned}\dot{x} &= u(x, y, z, t) \\ \dot{y} &= v(x, y, z, t) \\ \dot{z} &= w(x, y, z, t)\end{aligned}$$

where the dot denotes differentiation with respect to t in the Lagrangian form.

Conservation of Mass

1. Lagrange's Form

$$\frac{\dot{\rho}}{\rho \Delta} = 0 \quad \text{where} \quad \Delta = \frac{\partial(x, y, z)}{\partial(a, b, c)} \quad \text{i.e. the}$$

Jacobian of $x(a, b, c, t), y(a, b, c, t), z(a, b, c, t)$; a, b, c , being the above mentioned parameters.

2. Euler's Form

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

This may be written more simply in vector form,

$$\frac{\partial p}{\partial t} + \rho \nabla \cdot \mathbf{v} = 0$$

where the vector $\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$; $\mathbf{i}, \mathbf{j}, \mathbf{k}$ being unit vectors in the x, y, z , directions, respectively.

Conservation of Momentum

1. Lagrange's Form

$$\rho \ddot{x} + \frac{\partial p}{\partial x} = 0, \quad \rho \ddot{y} + \frac{\partial p}{\partial y} = 0, \quad \rho \ddot{z} + \frac{\partial p}{\partial z} = 0$$

which may be written as

$$\frac{d\mathbf{v}}{dt} + \frac{1}{\rho} \nabla p = 0$$

where $\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$. Another common way this is expressed is

$$\frac{d\mathbf{v}}{dt} + \nabla h = T \nabla s$$

$$\text{or} \quad \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2} \nabla v^2 - \mathbf{v} \times (\nabla \times \mathbf{v}) + \nabla h = T \nabla s$$

2. Euler's Form

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw) + \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2) + \frac{\partial}{\partial z}(\rho vw) + \frac{\partial p}{\partial y} = 0$$

$$\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial x}(\rho uw) + \frac{\partial}{\partial y}(\rho vw) + \frac{\partial}{\partial z}(\rho w^2) + \frac{\partial p}{\partial z} = 0$$

or, in vector form

$$\rho \frac{d\mathbf{v}}{dt} = \rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p$$

Conservation of Energy

1. Lagrange's form

$$\rho \dot{E}_{tot} + \frac{\partial(\rho \dot{x})}{\partial x} + \frac{\partial(\rho \dot{y})}{\partial y} + \frac{\partial(\rho \dot{z})}{\partial z} = 0$$

where $E_{tot} = E + \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$, E being internal energy per unit mass.

In vector notation, the above may be written as

$$\rho \frac{d}{dt} \left[E + \frac{1}{2} v^2 \right] + \nabla \cdot (\rho \mathbf{v}) = 0$$

This may be reduced to $\dot{S}=0$ by using $dE = v^2 p \, dp / T ds$, and the equations of conservation of mass and momentum.

2. Euler's Form

With Ψ defined as above, and $V^2 = u^2 + v^2 + w^2 = |\mathbf{v}|^2$

$$\frac{\rho}{2} \frac{\partial V^2}{\partial t} + \rho T \frac{\partial S}{\partial t} + \rho u \frac{\partial}{\partial x} \left(\frac{V^2}{2} + h \right) + \rho v \frac{\partial}{\partial y} \left(\frac{V^2}{2} + h \right) + \rho w \frac{\partial}{\partial z} \left(\frac{V^2}{2} + h \right) = 0$$

or

$$\frac{\partial}{\partial t} \left[\rho \left(\frac{V^2}{2} + E \right) \right] + \frac{\partial}{\partial x} \left[\rho u \left(\frac{V^2}{2} + h \right) \right] + \frac{\partial}{\partial y} \left[\rho v \left(\frac{V^2}{2} + h \right) \right] + \frac{\partial}{\partial z} \left[\rho w \left(\frac{V^2}{2} + h \right) \right] = 0$$

or, in vector form,

$$\frac{\partial}{\partial t} \left[\rho \left(\frac{V^2}{2} + E \right) \right] + \nabla \cdot \left[\rho \left(\frac{V^2}{2} + h \right) \mathbf{v} \right] = 0$$

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LIST OF SYMBOLS

a	acceleration
c	$\left(\frac{\partial p}{\partial \rho}\right)_s^{1/2}$
c _v	heat capacity at constant volume
f	a function
g	$1 - \frac{p}{U} \frac{dU}{dP}$
h	enthalpy
i	total impulse
K	ratio of heat capacities, or constant
l	thickness of shock front
m	mass
p	absolute pressure
r	Euler Coordinate
s	entropy
t	time
v	specific volume
u, v, w	velocity components
x, y, z	Cartesian coordinates
E	internal energy
G	a property, or $1 - \left(\frac{p_0 U}{\rho c}\right)^2$
H	enthalpy
F, J, M	constants
L, M, N	functions of P
P	over pressure, p-p ₀
R	Lagrange coordinate
T	absolute temperature
U	Shock wave velocity
W	flow velocity behind wave front, or charge weight
Z	$R/W^{1/3}$

ξ	$\frac{2\omega}{R} + 1$
λ	Heat transfer coefficient
μ	friction or viscosity coefficient
ξ	Thickness of fluid disk
π	$\frac{p_2}{p_1}$, or $\frac{p-p_0}{p_0}$
ρ	density
σ	$\int_{p_0}^p \frac{c}{p} dp$
τ	Reduced time, or a constant time

Subscript (o) denotes initial conditions or properties of the undisturbed fluid.